

INTRODUCTION TO QUANTUM FIELD THEORY

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Abstract

Even the uninitiated will know that Quantum Field Theory cannot be introduced systematically in just four lectures. I try to give a reasonably connected outline of part of it, from second quantization to the path-integral technique in Euclidean space, where there is an immediate connection with the rules for Feynman diagrams and the partition function of Statistical Mechanics.

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INTRODUCTION TO QUANTUM FIELD THEORY

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ABSTRACT

Even the uninitiated will know that Quantum Field Theory cannot be introduced systematically in just four lectures. I try to give a reasonably connected outline of part of it, from second quantization to the path-integral technique in Euclidean space, where there is an immediate connection with the rules for Feynman diagrams and the partition function of Statistical Mechanics.

1. Why Introduce Quantum Fields?

In ordinary quantum mechanics, displacement is an operator \mathbf{X} , but time t is just a parameter which labels Schrödinger state vectors or Heisenberg operators such as $\mathbf{X} = \mathbf{X}(t)$. This does not sit well with special relativity, which places displacement and time on the same footing.

If time were an operator T , it would be the component of a four-position operator $X^\mu = (T, \mathbf{X})$ conjugate to the Hamiltonian H in the four-momentum¹ $P^\mu = (H, \mathbf{P})$:

$$[P^\mu, X^\nu] = ig^{\mu\nu} \quad (1.1)$$

The commutator $[H, T] = i$ implies

$$\exp(-i\epsilon T) H \exp(i\epsilon T) = H - \epsilon \quad (1.2)$$

for any constant ϵ , so the operator $\exp(i\epsilon T)$ applied to any eigenstate $|E\rangle$ of H with energy eigenvalue E produces another eigenstate

$$\exp(i\epsilon T) |E\rangle$$

with shifted eigenvalue $E - \epsilon$. That indicates the presence of a continuous energy spectrum with range $-\infty < E < \infty$, contrary to the requirement that E be bounded below. Also, it contradicts the fact that generally, E is quantized [1].

So instead, we demote displacement to the status of parameter, like t . The dynamical quantities describing matter are to be operators labelled by a spatial three-vector \mathbf{x} as well as the time t . Such operators are *quantum fields*

$$\phi = \phi(x) \quad (1.3)$$

where x denotes the four-vector parameter (t, \mathbf{x}) .

¹Generally, we use natural units $\hbar = c = \epsilon_0 = 1$. The metric tensor $g^{\mu\nu}$ is $g^{00} = 1$, $g^{0i} = 0$, $g^{ij} = -\delta^{ij}$ for spatial indices $i, j = 1, 2, 3$. Thus the four-derivative $\partial^\mu = \partial/\partial x_\mu$ has components $(\partial/\partial t, -\nabla)$. The sign of the antisymmetric tensor $\varepsilon_{\alpha\beta\gamma\delta}$ is fixed by $\varepsilon_{0123} = +1 = -\varepsilon^{0123}$.

2. Photons

Experimental evidence for field quantization dates back to the discovery of photons. The success of Einstein's analyses of the photoelectric effect in 1905 and photon emission and absorption in 1917 [2] contradicted purely classical interpretations of the electromagnetic four-potential $A_\mu(x)$ and field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (2.1)$$

With the birth of quantum mechanics in 1925–26, it became evident that $F_{\mu\nu}$ and A_μ must be operators acting on state vectors, like other dynamical variables, with Maxwell's equations understood to be operator relations:

$$\partial^\mu F_{\mu\nu} = j_\nu \quad , \quad \partial^\mu \tilde{F}_{\mu\nu} = 0 \quad (2.2)$$

Here $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is the dual tensor of $F_{\alpha\beta}$, and $j_\mu(x)$ is the conserved four-current density due to charged matter ($\partial^\mu j_\mu = 0$).

Quantum electrodynamics began in 1927 with Dirac's paper [3] on the operator structure of the Hamiltonian. His method has become known as “second quantization”.

Consider photons within a large box of volume $V = L_1 \times L_2 \times L_3$. Periodic boundary conditions in each spatial dimension restrict the allowed wave vectors to values

$$\mathbf{k} = 2\pi (m_1/L_1, m_2/L_2, m_3/L_3) \quad (2.3)$$

where m_1, m_2, m_3 are integers. Continuously varying \mathbf{k} can be obtained in the limit $V \rightarrow \infty$.

According to Einstein, these photons have electromagnetic energy

$$\sum_{\mathbf{k}} \sum_{\lambda=1}^2 |\mathbf{k}| n_{\mathbf{k}\lambda}$$

where $n_{\mathbf{k}\lambda}$ is the number of photons with wave vector \mathbf{k} and transverse polarization λ , and $\sum_{\mathbf{k}}$ means sum over m_1, m_2, m_3 . Dirac understood the integers $n_{\mathbf{k}\lambda} \geq 0$ to be eigenvalues of a quantum number operator $N_{\mathbf{k}\lambda}$, and so obtained a Hamiltonian operator for photons:

$$H_{\text{photon}} = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 |\mathbf{k}| N_{\mathbf{k}\lambda} \quad (2.4)$$

How should the quantization of $N_{\mathbf{k}\lambda}$ be achieved? The operator $-i\partial/\partial\theta$ conjugate to an angle variable θ with period 2π has integer eigenvalues, but these run from $+\infty$ down to $-\infty$, contrary to the requirement that energy be bounded below.

Dirac observed that the required eigenvalue spectrum can be obtained from the algebra of raising and lowering operators for independent harmonic oscillators. Consider operators $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^\dagger$ with oscillator-like commutation relations

$$[a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\lambda\lambda'} \quad , \quad [a_{\mathbf{k}\lambda}, a_{\mathbf{k}'\lambda'}] = 0 = [a_{\mathbf{k}\lambda}^\dagger, a_{\mathbf{k}'\lambda'}^\dagger] \quad (2.5)$$

Then it is possible to make the identification

$$N_{\mathbf{k}\lambda} = a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} \quad (2.6)$$

because we get integer eigenvalues running from 0 to ∞ . The result for the photon Hamiltonian is a quadratic form

$$H_{\text{photon}} = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 |\mathbf{k}| a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} \quad (2.7)$$

with minimum eigenvalue zero corresponding to a state $|0\rangle$ containing no photons. Since $a_{\mathbf{k}\lambda}$ reduces the number of photons of type (\mathbf{k}, λ) by one, it is called a photon annihilation operator. Similarly, $a_{\mathbf{k}\lambda}^\dagger$ “creates” a photon.

The theory (2.7) describes free photons only, since it implies that all number operators are conserved:

$$[N_{\mathbf{k}\lambda}, H_{\text{photon}}] = 0 \quad (2.8)$$

Dirac included photon interactions by adding a linear combination of $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^\dagger$ to correspond to the absorption or emission of a single photon in first-order perturbation theory:

$$H_{\text{int}} = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 (V_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} + a_{\mathbf{k}\lambda}^\dagger V_{\mathbf{k}\lambda}) \quad (2.9)$$

The factors $V_{\mathbf{k}\lambda}$ and $V_{\mathbf{k}\lambda}^\dagger$ depend on the dynamical variables characterizing charged matter, such as atoms. The full Hamiltonian is then of the form

$$H = H_{\text{photon}} + H_{\text{atom}} + H_{\text{int}} \quad (2.10)$$

where H_{atom} describes the unperturbed atomic system.

This theory is not fully explicit — it specifies the dependence of H on $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^\dagger$ but not on the dynamical variables for charged matter. Nevertheless Dirac was able to derive from his theory the key points of Einstein’s analysis of photon emission and absorption, and hence the A and B coefficients. Briefly, the relations

$$a_{\mathbf{k}\lambda}^\dagger |\dots n_{\mathbf{k}\lambda} \dots\rangle = \sqrt{n_{\mathbf{k}\lambda} + 1} |\dots n_{\mathbf{k}\lambda} + 1 \dots\rangle \quad , \quad a_{\mathbf{k}\lambda} |\dots n_{\mathbf{k}\lambda} \dots\rangle = \sqrt{n_{\mathbf{k}\lambda}} |\dots n_{\mathbf{k}\lambda} - 1 \dots\rangle \quad (2.11)$$

imply that, given $n_{\mathbf{k}\lambda}$ photons initially, the probabilities of emission and absorption are given by

$$\mathcal{P}_{\text{em}} = (n_{\mathbf{k}\lambda} + 1) \mathcal{P}_{\mathbf{k}\lambda} \quad (2.12)$$

and

$$\mathcal{P}_{\text{abs}} = n_{\mathbf{k}\lambda} \mathcal{P}_{\mathbf{k}\lambda} \quad (2.13)$$

where $\mathcal{P}_{\mathbf{k}\lambda}$ is the emission probability in the absence of photons. Thus the emission probability consists of a spontaneous part $\mathcal{P}_{\mathbf{k}\lambda}$, plus an induced part $n_{\mathbf{k}\lambda} \mathcal{P}_{\mathbf{k}\lambda}$ equal to the absorption probability, in agreement with Einstein [2].

3. Electromagnetic Field Operators

The photon operators $a_{\mathbf{k}\lambda}$ and $a_{\mathbf{k}\lambda}^\dagger$ can be associated with the corresponding Fourier components and polarizations of $F^{\mu\nu}$ and A^μ if these fields are *free*. This means that charged matter is omitted in Maxwell's equations (2.2), at least in an initial approximation:

$$\partial^\mu F_{\mu\nu} = 0 \quad , \quad \partial^\mu \tilde{F}_{\mu\nu} = 0 \quad (3.1)$$

At this point, we have to deal with the partly unphysical nature of the four-potential A_μ . Photons are polarized transversely, i.e. in only two directions $\lambda = 1, 2$. The remaining ‘‘longitudinal’’ and ‘‘scalar’’ components of A_μ must be sidelined in some manner. This is done by exploiting gauge ambiguities in the four-potential:

$$A_\mu \longrightarrow A_\mu + \partial_\mu \xi \quad \text{for any } \xi = \xi(x) \quad (3.2)$$

These transformations are an exact invariance of electromagnetism, so they act within a completely unphysical sector of the theory.

3.1 Unphysical Components

There are various ways of proceeding, depending on taste². One is to use (3.1) and (3.2) to eliminate the two extra components of A_μ completely.

For each potential A_μ , consider the A_μ -dependent gauge transformation

$$A_\mu \longrightarrow \mathcal{A}_\mu(t, \mathbf{x}) = A_\mu(t, \mathbf{x}) - \frac{\partial}{\partial x^\mu} \int d^3y \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \nabla_{\mathbf{y}} \cdot \mathbf{A}(t, \mathbf{y}) \quad (3.3)$$

A shorthand for this is

$$\mathcal{A}_\mu = A_\mu + \partial_\mu \nabla^{-2} \nabla \cdot \mathbf{A} \quad (3.4)$$

where

$$\nabla^{-2} f(t, \mathbf{x}) = -\frac{1}{4\pi} \int d^3y \frac{f(t, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \quad (3.5)$$

defines the action of the inverse Laplacian ∇^{-2} on a function f . Convergence of (3.5) imposes a boundary condition on potentials at space-like infinity such that zero modes of ∇^{-2}

$$a_\mu(x) = x^\nu f_{\mu\nu}(x_0) \quad (3.6)$$

are excluded. Then both $\nabla^2 \nabla^{-2} = 1$ and $\nabla^{-2} \nabla^2 = 1$ are legitimate operator identities.

The new three-vector potential

$$\mathcal{A} = \mathbf{A} - \nabla \nabla^{-2} \nabla \cdot \mathbf{A} \quad (3.7)$$

obeys the Coulomb gauge condition

$$\nabla \cdot \mathcal{A} = 0 \quad (3.8)$$

²See Chapter 14 of Bjorken and Drell [4]. Lee [5] and Ryder [6] quantize directly in the Coulomb gauge. Bogoliubov and Shirkov [7], Schweber [8], Itzykson and Zuber [9], and Mandl and Shaw [10] use the Lorentz covariant procedure of Gupta and Bleuler [11], where Hilbert space is expanded to include unphysical states of zero and negative norm.

so it is transverse: its Fourier components are orthogonal to the wave vector \mathbf{k} . The new scalar potential is

$$\mathcal{A}_0 = A_0 + \partial_0 \nabla^{-2} \nabla \cdot \mathbf{A} \quad (3.9)$$

Eq. (3.1) contains Gauss's Law for free fields

$$\partial^\mu F_{\mu 0} = -\nabla^2 A_0 - \partial_0 \nabla \cdot \mathbf{A} = 0 \quad (3.10)$$

which requires the temporal gauge condition

$$\mathcal{A}_0 = 0 \quad (3.11)$$

to be obeyed as well.

The lack of Lorentz covariance of this procedure is only apparent, since A_μ is not a true four-vector — any Lorentz transformation $x \rightarrow x' = \Lambda x$ may be accompanied by a gauge transformation ξ_Λ with arbitrary dependence on Λ :

$$A_\mu(x) \longrightarrow A'_\mu(x') = \Lambda_\mu{}^\nu A_\nu(x) + \partial_\mu \xi_\Lambda(x) \quad (3.12)$$

More generally, let N_μ be any constant time-like vector with $N^2 = 1$, and let

$$((N \cdot \partial)^2 - \partial^2)^{-1} f(x) = - \int d^4 y \delta(N \cdot x - N \cdot y) f(y) / \sqrt{-16\pi^2(x-y)^2} \quad (3.13)$$

Then free fields

$$\mathcal{A}_\mu = A_\mu + \partial_\mu ((N \cdot \partial)^2 - \partial^2)^{-1} (\partial \cdot A - N \cdot \partial N \cdot A) \quad (3.14)$$

satisfy transversality conditions

$$\partial \cdot \mathcal{A} = 0, \quad N \cdot \mathcal{A} = 0 \quad (3.15)$$

The choice of N_μ has no physical significance: any two N 's can be related by a gauge transformation.

3.2 Fourier Decomposition

The field strengths $F_{\mu\nu}$ are gauge invariant, so we have

$$F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \quad (3.16)$$

Eqs. (3.15) and (3.16) imply the equation of motion

$$\partial^2 \mathcal{A}_\mu = 0 \quad (3.17)$$

and hence the Fourier decomposition

$$\mathcal{A}^\mu = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 \{ c_{\mathbf{k}\lambda} \varepsilon_{\mathbf{k}\lambda}^\mu e^{-ik \cdot x} + c_{\mathbf{k}\lambda}^\dagger \varepsilon_{\mathbf{k}\lambda}^{\mu*} e^{ik \cdot x} \} \quad (3.18)$$

in which the variable k^μ conjugate to x_μ lies on the forward light cone

$$k^\mu = (|\mathbf{k}|, \mathbf{k}) \quad (3.19)$$

and the four-vectors $\varepsilon_{\mathbf{k}\lambda}^\mu$ are orthogonal to both k_μ and N_μ :

$$k \cdot \varepsilon_{\mathbf{k}\lambda} = 0 = N \cdot \varepsilon_{\mathbf{k}\lambda} \quad (3.20)$$

For the moment, the coefficients $c_{\mathbf{k}\lambda}$ and $c_{\mathbf{k}\lambda}^\dagger$ remain unspecified, except that they are operators chosen to make \mathcal{A}_μ Hermitian.

The polarization vectors $\varepsilon_{\mathbf{k}\lambda}^\mu$ are space-like, and can be chosen orthogonal to each other:

$$\varepsilon_{\mathbf{k}\lambda} \cdot \varepsilon_{\mathbf{k}\lambda'}^* = -\delta_{\lambda\lambda'} \quad , \quad \lambda, \lambda' = 1, 2 \quad (3.21)$$

For example, if we take

$$N_\mu = (1, 0, 0, 0) \quad , \quad k_\mu = (k, 0, 0, k) \quad (3.22)$$

we can choose $\varepsilon_{\mathbf{k}\lambda}^\mu$ to be

$$\varepsilon_{\mathbf{k}x}^\mu = (0, 1, 0, 0) \quad , \quad \varepsilon_{\mathbf{k}y}^\mu = (0, 0, 1, 0) \quad (3.23)$$

for plane polarizations, or

$$\varepsilon_{\mathbf{k}\pm}^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0) \quad (3.24)$$

for circular polarizations. The values $\lambda = 1, 2$ refer to any orthogonal pair of polarizations, such as (3.23) or (3.24).

The set

$$\{N^\mu, N^\mu - k^\mu/k \cdot N, \varepsilon_{\mathbf{k}1}^\mu, \varepsilon_{\mathbf{k}2}^\mu\} \quad (3.25)$$

is orthonormal and complete in four-dimensional space with metric $g_{\mu\nu}$, so we have

$$g^{\mu\nu} = N^\mu N^\nu - (N^\mu - k^\mu/k \cdot N)(N^\nu - k^\nu/k \cdot N) - \sum_{\lambda=1}^2 \varepsilon_{\mathbf{k}\lambda}^\mu \varepsilon_{\mathbf{k}\lambda}^{\nu*} \quad (3.26)$$

and hence

$$\sum_{\lambda=1}^2 \varepsilon_{\mathbf{k}\lambda}^\mu \varepsilon_{\mathbf{k}\lambda}^{\nu*} = -g^{\mu\nu} + (N^\mu k^\nu + N^\nu k^\mu)/k \cdot N - k^\nu k^\nu / (k \cdot N)^2 \quad (3.27)$$

The presence of the unphysical entity N^μ in this formula is related to the lack of gauge invariance of the polarization vectors

$$\varepsilon_{\mathbf{k}\lambda}^\mu \longrightarrow \varepsilon_{\mathbf{k}\lambda}^\mu + k^\mu \xi_\lambda(k) \quad , \quad \lambda = 1, 2 \quad (3.28)$$

This occurs even though the label λ refers to transverse directions. Any calculation of a physical amplitude should be invariant under (3.28) and hence independent of artefacts like N_μ .

3.3 Connection with Photons

The summation variable \mathbf{k} in the Fourier expansion (3.18) of \mathcal{A}^μ is an obvious candidate for photon momentum, but the precise identification is not immediately clear. Does a photon of four-momentum k_μ correspond to a real wave

$$\sin(k \cdot x + \text{phase}) \quad (3.29)$$

as suggested by classical radiation theory, or to a complex de Broglie wave $e^{-ik \cdot x}$, and if the latter, what does $e^{+ik \cdot x}$ represent?

To resolve this question, consider a photon being absorbed by a heavy neutral system with charged constituents, such as an atom or a brick:

$$\{\text{atomic state } i\} + \gamma(\mathbf{k}, \lambda) \longrightarrow \{\text{atomic state } f\} \quad (3.30)$$

This process is induced by the Hamiltonian operator

$$H_{\text{int}}(t) = \int d^3x j^\mu(x) \mathcal{A}_\mu(x) \quad (3.31)$$

We consider first-order perturbations, so it is legitimate to use the free field \mathcal{A}_μ in (3.31). Also, we choose the temporal gauge (3.11). According to time-dependent perturbation theory [3, 12, 13], the amplitude for the transition (3.30) is

$$\mathcal{S}_{i+\gamma \rightarrow f} = -i \langle f | \int dt H_{\text{int}}(t) | i, \gamma \rangle = -i \int d^4x \langle f | \vec{j}(x) | i \rangle \cdot \langle 0 | \vec{\mathcal{A}}(x) | \gamma \rangle \quad (3.32)$$

If the system is sufficiently massive, both $|i\rangle$ and $|f\rangle$ can be represented by non-relativistic wave functions

$$\psi_i(t, \mathbf{x}) = e^{i(\mathbf{P}_i \cdot \mathbf{x} - E_i t)} \chi_i \quad (3.33)$$

and

$$\psi_f(t, \mathbf{x}) = e^{i(\mathbf{P}_f \cdot \mathbf{x} - E_f t)} \chi_f \quad (3.34)$$

where E_f, E_i and $\mathbf{P}_f, \mathbf{P}_i$ denote centre-of-mass energies and momenta. The spinors χ_i and χ_f depend on the internal dynamical variables to which \vec{j} couples. All dependence on the atomic centre-of-mass coordinate $x^\mu = (t, \mathbf{x})$ is carried by ψ_i and ψ_f^\dagger :

$$\langle f | \vec{j}(x) | i \rangle = e^{i(E_f - E_i)t} e^{-i(\mathbf{P}_f - \mathbf{P}_i) \cdot \mathbf{x}} \langle f | \vec{j}(0) | i \rangle \quad (3.35)$$

Consequently, the amplitude (3.32) can be written

$$\mathcal{S}_{i+\gamma \rightarrow f} = -i \int d^4x e^{i(E_f - E_i)t} e^{-i(\mathbf{P}_f - \mathbf{P}_i) \cdot \mathbf{x}} \langle 0 | \vec{\mathcal{A}}(x) | \gamma \rangle \cdot \langle f | \vec{j}(0) | i \rangle \quad (3.36)$$

Clearly, only one Fourier component in the formula (3.18) for \mathcal{A}^μ contributes to $\mathcal{S}_{i+\gamma \rightarrow f}$: the term proportional to $e^{-ik \cdot x}$ with

$$k^0 = |\mathbf{k}| = E_f - E_i, \quad \mathbf{k} = \mathbf{P}_f - \mathbf{P}_i \quad (3.37)$$

Therefore the amplitude to absorb a photon of type (\mathbf{k}, λ) is

$$\mathcal{S}_{i+\gamma \rightarrow f} = -(2\pi)^4 i \delta^4(P_f - P_i - k) f_{\mathbf{k}\lambda} \vec{\epsilon}_{\mathbf{k}\lambda} \cdot \langle f | \vec{j}(0) | i \rangle \quad (3.38)$$

where the amplitudes $f_{\mathbf{k}\lambda}$ are given by

$$f_{\mathbf{k}\lambda} = \langle 0 | c_{\mathbf{k}\lambda} | \gamma(\mathbf{k}, \lambda) \rangle \quad (3.39)$$

By absorbing phases into the definition of $|\gamma(\mathbf{k}, \lambda)\rangle$, we can require the c -numbers $f_{\mathbf{k}\lambda}$ to be real and positive.

Evidently the coefficient $c_{\mathbf{k}\lambda}$ in the Fourier series (3.18) is an operator which reduces photon number by one. For if the initial state had contained n photons, only that photon obeying the constraint (3.37) could have been absorbed in first-order perturbation theory, leaving $n - 1$ photons to continue on to the final state. Therefore $c_{\mathbf{k}\lambda}$ must be proportional to the photon annihilation operator $a_{\mathbf{k}\lambda}$:

$$c_{\mathbf{k}\lambda} = f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \quad (3.40)$$

The interacting photon transmits both orbital and spin angular momentum to the target atom. Generally, this is analyzed by decomposing $\vec{\mathcal{A}}(x)$ into multipole fields [12, 14, 15]. The simplest case is S -wave absorption, which dominates for inverse photon energy $|\mathbf{k}|^{-1}$ much larger than the dimensions of the target atom. A further simplification is to consider atomic transitions $|j, m_i\rangle \rightarrow |j, m_f\rangle$ in which the total angular momentum quantum number j of the atom is unchanged. Then the Wigner-Eckart theorem reduces to the projection theorem for vector operators [12, 13, 14]

$$\langle f | \vec{j} | i \rangle \propto \langle f | \vec{J} | i \rangle \quad (3.41)$$

where \vec{J} is the operator for total atomic angular momentum:

$$\mathcal{S}_{i+\gamma \rightarrow f} \propto \int d^4x e^{i(E_f - E_i)t} e^{-i(\mathbf{P}_f - \mathbf{P}_i) \cdot \mathbf{x}} \langle 0 | \vec{\mathcal{A}}(x) | \gamma \rangle \cdot \langle f | \vec{J} | i \rangle \quad (3.42)$$

In the frame (3.19), circular polarizations (3.24) give rise to raising and lowering operators for angular momentum:

$$\vec{\varepsilon}_{\mathbf{k}\pm} \cdot \vec{J} = \frac{1}{\sqrt{2}} (J_x \pm iJ_y) \quad (3.43)$$

We see that the absorption of a photon with polarization $\varepsilon_{\mathbf{k}+}^\mu$ increases the atomic spin component J_z by 1, so the photon is right-handed and carries spin (helicity) 1.

Photon emission can be analysed in the same way. The de Broglie wave $\varepsilon_{\mathbf{k}\lambda}^{\mu*} e^{ik \cdot x}$ in (3.18) corresponds to a photon of type (\mathbf{k}, λ) being emitted. Substituting (3.40), we find

$$\mathcal{A}^\mu = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 f_{\mathbf{k}\lambda} (a_{\mathbf{k}\lambda} \varepsilon_{\mathbf{k}\lambda}^\mu e^{-ik \cdot x} + a_{\mathbf{k}\lambda}^\dagger \varepsilon_{\mathbf{k}\lambda}^{\mu*} e^{ik \cdot x}) \quad (3.44)$$

Note the consistency between this result and Eqs. (2.9) and (3.31) for H_{int} .

3.4 Electromagnetic Energy and Momentum

The constants $f_{\mathbf{k}\lambda}$ in (3.44) can be determined by comparing Dirac's photon energy operator (2.7) with Maxwell's formula for electromagnetic energy

$$H_{\text{emag}} = \frac{1}{2} \int d^3x (\mathbf{E}^2 + \mathbf{B}^2) \quad (3.45)$$

where \mathbf{E} and \mathbf{B} are the electric and magnetic fields.

Eqs. (3.16) and (3.44) imply

$$\begin{aligned}
E_i &= F_{0i} = -i \sum_{\mathbf{k}} \sum_{\lambda=1}^2 f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} (k_0 \varepsilon_i - k_i \varepsilon_0)_{\mathbf{k}\lambda} e^{-ik \cdot x} + \text{h.c.} \\
B_i &= -\frac{1}{2} \varepsilon_{ijk} F_{jk} = i \sum_{\mathbf{k}} \sum_{\lambda=1}^2 f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} \varepsilon_{ilm} (k_l \varepsilon_m)_{\mathbf{k}\lambda} e^{-ik \cdot x} + \text{h.c.}
\end{aligned} \tag{3.46}$$

where ‘‘h.c.’’ denotes Hermitian conjugate, so $\int d^3x \mathbf{E}^2$ and $\int d^3x \mathbf{B}^2$ are sums $\sum_{\mathbf{k}\mathbf{k}'} \sum_{\lambda\lambda'}$ containing integrals

$$\int d^3x e^{ik \cdot x} e^{\mp ik' \cdot x} = V \delta_{\mathbf{k}, \pm \mathbf{k}'} \tag{3.47}$$

For $\mathbf{k}' = -\mathbf{k}$ contributions, a convention is needed to fix relative phases in $\varepsilon_{\pm \mathbf{k}\lambda}^\mu$ and hence $a_{\pm \mathbf{k}\lambda}$. In terms of circular polarizations $\lambda = \pm$, a simple choice is

$$\varepsilon_{-\mathbf{k}\pm}^\mu = \varepsilon_{\mathbf{k}\pm}^{\mu*} \tag{3.48}$$

Then Eqs. (3.19)–(3.21) imply

$$\begin{aligned}
\int d^3x \mathbf{E}^2 &= V \sum_{\mathbf{k}} \sum_{\lambda=\pm} k_0^2 f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} (f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger - f_{-\mathbf{k}\lambda} a_{-\mathbf{k}\lambda} e^{-2ik_0 t}) + \text{h.c.} \\
\int d^3x \mathbf{B}^2 &= V \sum_{\mathbf{k}} \sum_{\lambda=\pm} k_0^2 f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda} (f_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger + f_{-\mathbf{k}\lambda} a_{-\mathbf{k}\lambda} e^{-2ik_0 t}) + \text{h.c.}
\end{aligned} \tag{3.49}$$

The $\mathbf{k}' = -\mathbf{k}$ terms depend on time and do not conserve photon number, but, as might be expected, they cancel in the combination (3.45):

$$\begin{aligned}
H_{\text{emag}} &= V \sum_{\mathbf{k}} \sum_{\lambda=\pm} (k_0 f_{\mathbf{k}\lambda})^2 (a_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger + a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda}) \\
&= 2V \sum_{\mathbf{k}} \sum_{\lambda=\pm} (k_0 f_{\mathbf{k}\lambda})^2 (a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} + \frac{1}{2})
\end{aligned} \tag{3.50}$$

Here the labels $\lambda = 1, 2$ refer to any pair of transverse polarizations obeying Eq. (3.21).

The constant $\frac{1}{2}$ added to $a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda}$ in (3.50) is a consequence of the commutation relations (2.5). It is therefore not surprising that Dirac’s quantum formula (2.7) and the classical expression (3.45) should differ by such terms. Agreement in the classical limit $n_{\mathbf{k}\lambda} \gg 1$ is achieved by equating the coefficients of $a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda}$:

$$f_{\mathbf{k}\lambda} = \frac{1}{\sqrt{2k_0 V}}, \quad k_0 = |\mathbf{k}| \tag{3.51}$$

For quantum mechanical purposes, the constant term in H_{emag} must be subtracted off so that the no-photon state $|0\rangle$ has zero energy. The result is (of course) Dirac’s Hamiltonian

$$H_{\text{photon}} = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 k_0 a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} = H_{\text{emag}} - \sum_{\mathbf{k}} \sum_{\lambda=1}^2 \frac{1}{2} k_0 \tag{3.52}$$

This formula is often written in *normal ordered* form

$$H_{\text{photon}} = \frac{1}{2} \int d^3x : \mathbf{E}^2 + \mathbf{B}^2 : \quad (3.53)$$

where the symbol $:\dots:$ is an instruction to order all products such that creation operators appear to the left of all annihilation operators.

The subtraction

$$\sum_{\mathbf{k}} \sum_{\lambda=1}^2 \frac{1}{2} k_0 \quad (3.54)$$

is an infinite zero-point energy produced by the oscillator-like formalism (2.5). Such infinite ambiguities arise frequently in quantum field theory. They are a consequence of attempts to multiply field operators such as $\mathbf{E}(x)$ and $\mathbf{B}(x)$ evaluated at the *same* space-time point x . Normal ordering is a special case [42] of a general procedure known as *renormalization* in which each ambiguity is eliminated by imposing a common-sense physical requirement — in the case above, that $|0\rangle$ should have zero energy.

Eq. (3.51) completes the specification of the operator structure of the free field (3.44):

$$\mathcal{A}^\mu = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 \frac{1}{\sqrt{2k_0V}} \{ a_{\mathbf{k}\lambda} \varepsilon_{\mathbf{k}\lambda}^\mu e^{-ik \cdot x} + a_{\mathbf{k}\lambda}^\dagger \varepsilon_{\mathbf{k}\lambda}^{\mu*} e^{ik \cdot x} \} \quad (3.55)$$

We can check Eq. (3.55) by calculating the Poynting vector $\mathbf{E} \times \mathbf{B}$ and relating it to the three-momentum operator \mathbf{P} :

$$\mathbf{P} = \int d^3x : \mathbf{E} \times \mathbf{B} : = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 \mathbf{k} a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} \quad (3.56)$$

Here the $\mathbf{k}' = -\mathbf{k}$ terms sum to zero because

$$-\frac{1}{2} \sum_{\mathbf{k}} \sum_{\lambda=1}^2 a_{\mathbf{k}\lambda} a_{-\mathbf{k}\lambda} \mathbf{k} e^{-2ik_0 t} + \text{h.c.} \quad (3.57)$$

changes sign for $\mathbf{k} \rightarrow -\mathbf{k}$. The same argument is not really adequate for the ill-defined commutator term

$$\sum_{\mathbf{k}} \sum_{\lambda=1}^2 \frac{1}{2} \mathbf{k} \quad (3.58)$$

so normal ordering is required in Eq. (3.56).

The total four-momentum operator

$$P^\mu = \sum_{\mathbf{k}} \sum_{\lambda=1}^2 k^\mu a_{\mathbf{k}\lambda}^\dagger a_{\mathbf{k}\lambda} \quad (3.59)$$

satisfies the following commutation relations,

$$[P^\mu, a_{\mathbf{k}\lambda}] = -k^\mu a_{\mathbf{k}\lambda} \quad , \quad [P^\mu, a_{\mathbf{k}\lambda}^\dagger] = k^\mu a_{\mathbf{k}\lambda}^\dagger \quad (3.60)$$

so its commutator with $\mathcal{A}^\nu(x)$ corresponds to an infinitesimal displacement in space-time:

$$i[P^\mu, \mathcal{A}^\nu] = \partial^\mu \mathcal{A}^\nu \quad (3.61)$$

Finite displacements $x_\mu \rightarrow x_\mu + a_\mu$ can be obtained via a unitary transformation:

$$e^{iP \cdot a} \mathcal{A}^\mu(x) e^{-iP \cdot a} = \mathcal{A}^\mu(x + a) \quad (3.62)$$

3.5 Classical Waves

The free field $\mathcal{A}_\mu(x)$ given by Eq. (3.55) is obviously not diagonal in photon number. For example, it connects one-photon states to $|0\rangle$, but there is no expectation value for a single photon:

$$\langle \gamma | \mathcal{A}_\mu | \gamma \rangle = 0 \quad (3.63)$$

Indeed, we have seen that single photons are associated with de Broglie waves ($e^{ik \cdot x}$ for emission, $e^{-ik \cdot x}$ for absorption) and not with the real waves (3.29) of classical electromagnetism.

Real waves can be obtained as field expectation values formed with *coherent states* [9, 17]

$$|\eta_{\mathbf{k}\lambda}\rangle = \exp\left(-\frac{1}{2}|\eta_{\mathbf{k}\lambda}|^2\right) \exp(\eta_{\mathbf{k}\lambda} a_{\mathbf{k}\lambda}^\dagger) |0\rangle \quad (3.64)$$

which are eigenstates of $a_{\mathbf{k}\lambda}$:

$$a_{\mathbf{k}\lambda} |\eta_{\mathbf{k}\lambda}\rangle = \eta_{\mathbf{k}\lambda} |\eta_{\mathbf{k}\lambda}\rangle \quad (3.65)$$

Since $a_{\mathbf{k}\lambda}$ is not self-adjoint, its eigenvalues $\eta_{\mathbf{k}\lambda}$ can be complex and its eigenstates non-orthogonal:

$$\langle \eta_{\mathbf{k}\lambda} | \eta'_{\mathbf{k}\lambda} \rangle = \exp\left(i \operatorname{Im}(\eta_{\mathbf{k}\lambda}^* \eta'_{\mathbf{k}\lambda}) - \frac{1}{2}|\eta_{\mathbf{k}\lambda} - \eta'_{\mathbf{k}\lambda}|^2\right) \quad (3.66)$$

However, states with different (\mathbf{k}, λ) values are orthogonal:

$$\langle \eta_{\mathbf{k}\lambda} | \eta_{\mathbf{k}'\lambda'} \rangle = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\lambda\lambda'} \quad (3.67)$$

The expectation value of \mathcal{A}_μ can be obtained directly from Eq. (3.65):

$$\langle \eta_{\mathbf{k}\lambda} | \mathcal{A}^\mu(x) | \eta_{\mathbf{k}\lambda} \rangle = \sqrt{\frac{2}{k_0 V}} \operatorname{Re} \left\{ \eta_{\mathbf{k}\lambda} \varepsilon_{\mathbf{k}\lambda}^\mu e^{-ik \cdot x} \right\} \quad (3.68)$$

These real Fourier components may be superposed by considering expectation values $\langle \zeta, \eta | \mathcal{A}^\mu | \zeta, \eta \rangle$ formed with states

$$|\zeta, \eta\rangle = \sum_{\mathbf{k}\lambda} \zeta_{\mathbf{k}\lambda} |\eta_{\mathbf{k}\lambda}\rangle, \quad \sum_{\mathbf{k}\lambda} |\zeta_{\mathbf{k}\lambda}|^2 = 1 \quad (3.69)$$

3.6 Remarks

So far, we have treated the wave vector \mathbf{k} as a discrete variable (2.3). As the quantization volume becomes infinite, sums and Kronecker deltas are replaced by integrals and Dirac delta functions:

$$\begin{aligned} (2k_0 V)^{-1} \sum_{\mathbf{k}} &\longrightarrow \int \frac{d^3 k}{(2\pi)^3 2k_0}, & k_0 = |\mathbf{k}| \\ 2k_0 V \delta_{\mathbf{k}\mathbf{k}'} &\longrightarrow (2\pi)^3 2k_0 \delta^3(\mathbf{k} - \mathbf{k}') \end{aligned} \quad (3.70)$$

The extra factors $2k_0$ are conventional; they make the continuum result Lorentz invariant, as is evident from the identity

$$d^3k/2\sqrt{\mathbf{k}^2 + m^2} = d^4k \delta(k^2 - m^2)\theta(k_0) \quad (3.71)$$

with $m = 0$ for photons.

From Eq. (2.5), we see that the $V \rightarrow \infty$ limit for creation and annihilation operators is given by

$$(2k_0V)^{-1/2}a_{\mathbf{k}\lambda} \longrightarrow a_\lambda(\mathbf{k}) \quad (3.72)$$

where the new operators satisfy

$$[a_\lambda(\mathbf{k}), a_{\lambda'}(\mathbf{k}')] = 0 = [a_\lambda^\dagger(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')] \quad , \quad [a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')] = (2\pi)^3 2k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'} \quad (3.73)$$

The right-hand side of Eq. (3.73) corresponds to the normalization

$$\langle \gamma(\mathbf{k}, \lambda) | \gamma(\mathbf{k}', \lambda') \rangle = (2\pi)^3 2k_0 \delta^3(\mathbf{k} - \mathbf{k}') \delta_{\lambda\lambda'} \quad (3.74)$$

for one-photon states. Photon polarization vectors $\varepsilon_{\mathbf{k}\lambda}^\mu$ are unaffected by the limiting procedure.

Most discrete- \mathbf{k} formulas may be readily converted by applying these rules. For example, Eqs. (3.55) and (3.59) for the free field $\mathcal{A}^\mu(x)$ and the four-momentum operator P^μ become

$$\mathcal{A}^\mu(x) = \sum_{\lambda=1}^2 \int \frac{d^3k}{(2\pi)^3 2k_0} \{ a_\lambda(\mathbf{k}) \varepsilon_{\mathbf{k}\lambda}^\mu e^{-ik \cdot x} + a_\lambda^\dagger(\mathbf{k}) \varepsilon_{\mathbf{k}\lambda}^{\mu*} e^{ik \cdot x} \} \quad (3.75)$$

and

$$P^\mu = \sum_{\lambda=1}^2 \int \frac{d^3k}{(2\pi)^3 2k_0} a_\lambda^\dagger(\mathbf{k}) a_\lambda(\mathbf{k}) k^\mu \quad (3.76)$$

It is important to realise that equations such as (3.75) and (3.76) are valid only for free fields. An interacting four-potential A^μ satisfies

$$(\partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu) A^\mu = j_\nu \quad (3.77)$$

instead of Eq. (3.17), so it has Fourier components which do *not* satisfy the constraint $k^2 = 0$. Consequently, its Fourier coefficients are not simply creation and annihilation operators for single photons — they can create or destroy multiparticle states.

However, the translation property (3.62) is generally satisfied by field operators in local quantum field theories.

4. Field Commutators and Uncertainty

The field-strength tensor $F_{\mu\nu}$ is an operator, so its components need not and in general do not commute. Consequently, measurements of electric and magnetic

fields are governed by uncertainty relations which (for the moment) we write in the naive form

$$\Delta F_{\mu\nu}(x) \Delta F_{\alpha\beta}(y) \geq \frac{1}{2} |\langle [F_{\mu\nu}(x), F_{\alpha\beta}(y)] \rangle| \quad (4.1)$$

If the points x and y are space-like separated, these measurements should not interfere with each other — otherwise, it would be possible to transmit signals acausally. Therefore it is important that measurable fields like $F_{\mu\nu}$ should obey *microcausality conditions*

$$[F_{\mu\nu}(x), F_{\alpha\beta}(y)] = 0, \quad (x - y)^2 < 0 \quad (4.2)$$

As we shall see, the result for free fields satisfies Eq. (4.2), and illustrates another general characteristic of field commutators — that they are *singular* on the light cone $(x - y)^2 = 0$.

4.1 Free Fields

The free four-potential \mathcal{A}_μ given by Eq. (3.75) is a linear combination of operators $a_\lambda(\mathbf{k})$ and $a_\lambda^\dagger(\mathbf{k})$ with commutators given by Eq. (3.73). Therefore the field commutators for $\mathcal{A}^\mu(x)$ are

$$[\mathcal{A}^\mu(x), \mathcal{A}^\nu(y)] = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1}^2 \varepsilon_{\mathbf{k}\lambda}^\mu \varepsilon_{\mathbf{k}\lambda}^{\nu*} e^{-ik \cdot (x-y)} - \text{h.c.} \quad (4.3)$$

According to Eq. (3.27), the sum over polarizations is $-g^{\mu\nu}$ plus gauge terms depending on k^μ and N^μ , so we find

$$[\mathcal{A}^\mu(x), \mathcal{A}^\nu(y)] = -ig^{\mu\nu} \{\Delta^+(x-y) - \Delta^+(y-x)\} + \text{gauge terms} \quad (4.4)$$

where the notation

$$\Delta^+(x) = -i \int \frac{d^3k}{(2\pi)^3 2k_0} e^{-ik \cdot x}, \quad k_0 = |\mathbf{k}| \quad (4.5)$$

is standard [8, 9, 10]; the superscript + refers to the positive-energy projector $\theta(k_0)$ in Eq. (3.71).

As written, the integral (4.5) is not obviously convergent — it oscillates for $|\mathbf{k}|$ large. A similar problem is encountered for the one-dimensional Fourier transform of the Heaviside step function

$$\theta(k) = \begin{cases} 1 & \text{if } k > 0 \\ 0 & \text{if } k < 0 \end{cases} \quad (4.6)$$

Oscillations in $\int dk \theta(k) e^{-ik \cdot x}$ at $k = +\infty$ are controlled by substituting $x \rightarrow x - i\epsilon$ for positive ϵ and taking the limit $\epsilon \rightarrow 0$:

$$\int dk \theta(k) e^{-ik \cdot x} = \lim_{\epsilon \rightarrow 0} \int_0^\infty dk e^{-ik(x-i\epsilon)} = -i \lim_{\epsilon \rightarrow 0} (x - i\epsilon)^{-1} \quad (4.7)$$

Usually this is written $-i/(x - i\epsilon)$ with the limiting procedure understood, as in the formula

$$(x - i\epsilon)^{-1} - (x + i\epsilon)^{-1} = 2\pi i \delta(x) \quad (4.8)$$

Similarly, let the four-vector x^μ in Eq. (4.5) become complex:

$$x^\mu \rightarrow x^\mu - i\eta^\mu, \quad \eta^2 > 0, \quad \eta_0 > 0 \quad (4.9)$$

The four-vector η^μ is restricted to lie within the forward light cone to ensure convergence of the integral

$$\Delta^+(x - i\eta) = -i \int \frac{d^3k}{(2\pi)^3 2k_0} e^{-ik \cdot (x - i\eta)} \quad (4.10)$$

To calculate it, choose the frame $\eta^\mu = (\epsilon, 0, 0, 0)$, and introduce polar coordinates $(|\mathbf{k}|, \theta, \phi)$ for \mathbf{k} with the Z-axis along \mathbf{x} :

$$\Delta^+(x - i\eta) = -\frac{i}{8\pi^2} \int_0^\infty d|\mathbf{k}| |\mathbf{k}| e^{-i|\mathbf{k}|(x_0 - i\epsilon)} \int_{-1}^1 d(\cos \theta) e^{i|\mathbf{k}||\mathbf{x}| \cos \theta} = \frac{i}{4\pi^2 (x - i\eta)^2} \quad (4.11)$$

Then the expression (4.5) is given by the limit $\eta^\mu \rightarrow 0$, i.e.

$$\Delta^+(x) = \frac{i}{4\pi^2 (x^2 - i\epsilon x_0)} \quad (4.12)$$

in conventional notation. Notice that $\partial^2 \Delta^+(x - i\eta)$ vanishes for all η within the forward light cone, so the same is true for the boundary value (4.12):

$$\partial^2 \Delta^+(x) = 0 \quad (4.13)$$

The commutator (4.4) involves the Jordan-Pauli function [18]

$$\Delta(x) = \Delta^+(x) - \Delta^+(-x) \quad (4.14)$$

From Eqs. (4.8) and (4.12), we find

$$\Delta(x) = \frac{i}{4\pi^2} \left\{ \frac{1}{x^2 - i\epsilon x_0} - \frac{1}{x^2 + i\epsilon x_0} \right\} = -\frac{1}{2\pi} \varepsilon(x_0) \delta(x^2) \quad (4.15)$$

where $\varepsilon(t)$ takes values ± 1 according to the sign of t :

$$\varepsilon(t) = t/|t| = \theta(t) - \theta(-t) \quad (4.16)$$

The gauge terms in (4.4) correspond to the formal substitution $k_\mu \rightarrow i\partial_\mu$ in the polarization sum (3.27):

$$[\mathcal{A}^\mu(x), \mathcal{A}^\nu(y)] = i \left\{ -g^{\mu\nu} + (N^\mu \partial^\nu + N^\nu \partial^\mu) (N \cdot \partial)^{-1} - \partial^\mu \partial^\nu (N \cdot \partial)^{-2} \right\} \Delta(x - y) \quad (4.17)$$

Some caution is necessary here because the requirements

$$N \cdot \partial (N \cdot \partial)^{-1} = 1 = (N \cdot \partial)^{-1} N \cdot \partial \quad (4.18)$$

do not fix $(N \cdot \partial)^{-1}$ uniquely — the pole $(N \cdot k \pm i\epsilon)^{-1}$ in Fourier space can be specified in various ways.

For example, the notation ∂_0^{-1} could refer to either of the prescriptions

$$(\partial_0 \pm \epsilon)^{-1} f(t, \mathbf{x}) = \int_{\mp\infty}^t dt' f(t', \mathbf{x}) \quad (4.19)$$

or a mixture thereof. However the ambiguity disappears if $f(t, \mathbf{x})$ can be written in the form

$$f(t, \mathbf{x}) = F(t, \mathbf{x}) - F(t - c, \mathbf{x}) , \quad c = \text{constant} \quad (4.20)$$

We find uniquely:

$$\partial_0^{-1} f(t, \mathbf{x}) = \int_{t-c}^t dt' F(t', \mathbf{x}) \quad (4.21)$$

The commutator function $\Delta(x)$ is of the form (4.20),

$$\Delta(x) = \frac{1}{4\pi|\mathbf{x}|} (\delta(t + |\mathbf{x}|) - \delta(t - |\mathbf{x}|)) \quad (4.22)$$

so we can calculate

$$\partial_0^{-1} \Delta(x) = \frac{1}{4\pi|\mathbf{x}|} (\theta(t + |\mathbf{x}|) - \theta(t - |\mathbf{x}|)) \quad (4.23)$$

and

$$\partial_0^{-2} \Delta(x) = \frac{1}{8\pi|\mathbf{x}|} (|t + |\mathbf{x}|| - |t - |\mathbf{x}||) \quad (4.24)$$

uniquely. A check is to calculate the inverse Laplacian (3.5) of $\Delta(x)$ and obtain

$$\nabla^{-2} \Delta(x) = \partial_0^{-2} \Delta(x) \quad (4.25)$$

in agreement with the property $\partial^2 \Delta(x) = 0$ implied by Eqs. (4.13) and (4.14). As a result, the non-vanishing commutators (4.17) for the case $N^\mu = (1, 0, 0, 0)$ can be written

$$[\mathcal{A}_i(x), \mathcal{A}_j(y)] = i (\delta_{ij} - \nabla_i \nabla_j \nabla^{-2}) \Delta(x - y) , \quad i, j = 1, 2, 3 \quad (4.26)$$

The generalization of this discussion to other timelike vectors N^μ is tedious but straightforward.

The gauge terms in Eq. (4.17) are total derivatives, so it follows from Eq. (3.16) that they do not contribute to $F_{\mu\nu}$ commutators. Consequently, the result for free fields is [18]

$$[F_{\mu\nu}(x), F_{\alpha\beta}(y)] = \frac{i}{2\pi} (\partial_\mu \delta_\nu^\sigma - \partial_\nu \delta_\mu^\sigma)_x (\partial_\alpha g_{\sigma\beta} - \partial_\beta g_{\sigma\alpha})_y \{ \varepsilon(x_0 - y_0) \delta((x - y)^2) \} \quad (4.27)$$

As promised, the result vanishes for space-like separations and is singular on the light cone.

Eq. (4.27) has special features not characteristic of the interacting case:

- (a) It is conserved in each of the indicies μ, ν, α, β . This is required by the free-field equations (3.1), but cannot be maintained in the presence of the current operator j_ν in Maxwell's equations (2.2).
- (b) It is proportional to the unit operator I .
- (c) It vanishes for time-like separations. This property is peculiar to massless free fields.

Notice that the gauge terms in Eqs. (4.17) and (4.26) do *not* vanish at spacelike separations. This becomes clear when Eq. (4.24) is written

$$\partial_0^{-2}\Delta(x) = \frac{t}{4\pi|\mathbf{x}|}\theta(-x^2) + \frac{1}{4\pi}\varepsilon(t)\theta(x^2) \quad (4.28)$$

Of course, these effects are not physical, so microcausality is not in question.

4.2 Measuring Electric and Magnetic Fields

At first sight, the light-cone singularity of the commutator (4.27) seems to be a problem. If the right-hand side of Eq. (4.1) can be infinite, are the components of $F_{\mu\nu}$ measurable in any sense?

Bohr and Rosenfeld [19] observed that the description of \mathbf{E} and \mathbf{B} as field components at each space-time point is an idealization of the actual physical situation. Observed quantities are really averages of these field components over various space-time regions. In other words, one can discuss the observed values of $F_{\mu\nu}$ components for a *neighbourhood* of any space-time point, but not the value at such a point.

In modern terminology [20, 21, 22], the operators $\mathbf{E}(x)$ and $\mathbf{B}(x)$ are *generalized functions*, or *distributions*: they represent the set of smeared operators

$$F_{\mu\nu}[f] = \int d^4x f(x)F_{\mu\nu}(x) \quad (4.29)$$

corresponding to all smooth functions $f(x)$ decreasing rapidly at $x_\mu \sim \infty$ (faster than any inverse power). A familiar classical example of this is the charge density $\rho(\mathbf{x})$ of a point charge Q at \mathbf{a} :

$$\rho(\mathbf{x}) = Q\delta^3(\mathbf{x} - \mathbf{a}) \quad (4.30)$$

Eq. (4.30) really refers to the linear functional ρ which assigns the number $Qg(\mathbf{a})$ to each “test” function $g = g(\mathbf{x})$:

$$g(\mathbf{x}) \xrightarrow{\rho} Qg(\mathbf{a}) = \rho[g] \quad (4.31)$$

A similar interpretation is to be given to the boundary values of complex functions, such as Eqs. (4.7) and (4.12). Note that smearing for $F_{\mu\nu}(x)$ and associated distributions like $\Delta(x)$ is over *both* space and time.

Smearing becomes essential if we are to have a sensible interpretation of the uncertainty $\Delta F_{\mu\nu}$. Without smearing, $\Delta F_{\mu\nu}(x)$ would have to be represented algebraically as follows,

$$\sqrt{\langle F_{\mu\nu}(x)F_{\mu\nu}(x) - \langle F_{\mu\nu}(x) \rangle^2 \rangle} \quad (4.32)$$

with no summation over μ, ν . The trouble with this formula is that operators are multiplied together at the same space-time point x_μ . It is clear from the commutator (4.27) or the discussion of normal ordering below Eq. (3.53) that such an expression is infinite, by construction! Instead, we should consider the uncertainty in $F_{\mu\nu}[f]$ for each test function f :

$$\Delta F_{\mu\nu}[f] = \sqrt{\langle F_{\mu\nu}[f]^2 - \langle F_{\mu\nu}[f] \rangle^2 \rangle}, \quad \text{no sum over } \mu, \nu \quad (4.33)$$

Then the uncertainty relations

$$\Delta F_{\mu\nu}[f] \Delta F_{\alpha\beta}[g] \geq \frac{1}{2} |\langle [F_{\mu\nu}[f], F_{\alpha\beta}[g]] \rangle| \quad (4.34)$$

make perfect sense when the free-field result (4.27) is substituted.

A less formal version of this conclusion is that infinite results will not be observed because the light-cone singularity in Eq. (4.27) is *integrable*. This crucial observation was the starting point for the work of Bohr and Rosenfeld [19]. After a long analysis of carefully constructed thought experiments, they managed to show how the constraints implied by Eqs. (4.27) and (4.34) can be deduced from the uncertainty relations

$$(\Delta p \Delta x)_{\text{probe}} \geq \frac{1}{2} \hbar \quad (4.35)$$

obeyed by any probe used to measure electromagnetic field strengths. The discussion is not easily summarized, as several reviewers [23] have discovered.

An important consequence of the Bohr-Rosenfeld analysis is that a field such as A_μ coupled causally to quantized matter cannot remain classical — it must itself be quantized.

5. Local Quantum Field Theories

As we saw in Section 1, quantum fields are needed to describe all relativistic matter, not just photons. So we consider a collection of fields

$$\phi_j = \phi_j(x) \quad (5.1)$$

The subscript j refers to spin, charge or other distinguishing quantum numbers carried by each field component.

5.1 General Properties

In local theories, all field components are assumed to depend on a single³ space-time variable x^μ , such that active translations of states through a space-time interval a^μ

$$|\psi\rangle \longrightarrow |\psi_a\rangle = e^{-iP \cdot a} |\psi\rangle \quad (5.2)$$

correspond to a shift $x^\mu \rightarrow x^\mu + a^\mu$ in the space-time dependence of these operators:

$$\langle \xi_a | \phi_j(x+a) | \psi_a \rangle = \langle \xi | \phi_j(x) | \psi \rangle \quad (5.3)$$

Eq. (5.3) holds for all states $|\psi\rangle$ and $|\xi\rangle$, so the translation property previously noted for the free photon field in Eq. (3.62) is obtained as a general result:

$$\phi_j(x+a) = e^{iP \cdot a} \phi_j(x) e^{-iP \cdot a} \quad (5.4)$$

³String theories [24] generalize local field theory. They involve operators depending on a line of space-time points (string). By Taylor expansion, each operator of this type can be regarded as equivalent to an infinite set of local field operators.

A similar property holds for Lorentz transformations

$$x \longrightarrow x' = \Lambda x \quad (5.5)$$

and the corresponding unitary operator U_Λ . The new feature is that those indices j of ϕ_j labelling spin components are mixed:

$$U_\Lambda \phi_j(x) U_\Lambda^\dagger = \phi_i(\Lambda x) \mathcal{D}_{ij}(\Lambda) \quad (5.6)$$

Matrices $\mathcal{D}(\Lambda)$ constructed from the coefficients $\mathcal{D}_{ij}(\Lambda)$ form a representation of the Lorentz group:

$$\mathcal{D}(\Lambda_1 \Lambda_2) = \mathcal{D}(\Lambda_1) \mathcal{D}(\Lambda_2) \quad (5.7)$$

Another general property is microcausality, already considered for the electromagnetic field strength in Eq. (4.2). Here it is necessary to distinguish bosonic and fermionic field components. Consider components $\phi_i[f]$ and $\phi_j[g]$ smeared over functions $f(x)$ and $g(x)$ with space-like separated “supports” (regions in which the functions are non-zero almost everywhere). According to Fermi-Dirac statistics, the state

$$|i, f; j, g; \psi\rangle = \phi_i[f] \phi_j[g] |\psi\rangle \quad (5.8)$$

should be anti-symmetric under the interchange

$$\left\{ \begin{array}{c} i \\ f \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} j \\ g \end{array} \right\}$$

if both ϕ_i and ϕ_j are fermionic; otherwise, the interchange should be symmetric⁴. So the general form of the microcausality condition is

$$[\phi_i(x), \phi_j(y)]_\mp = \phi_i(x) \phi_j(y) \mp \phi_j(y) \phi_i(x) = 0, \quad (x - y)^2 < 0 \quad (5.9)$$

where the anti-commutator $[\ ,]_+$ is used only if both ϕ_1 and ϕ_2 are fermionic. Note that the anti-commutator version of (5.9) does not contradict the independence of measurements at space-like separations because such measurements are possible only for *even* powers of fermionic operators.

The book of Streater and Wightman [20] analyses these axiomatic requirements in a manner which is advanced and rigorous but nevertheless instructive. The most important results of axiomatic field theory are that:

- (a) Field components ϕ_i are bosonic or fermionic according to whether the Lorentz representation $\mathcal{D}(\Lambda)$ in Eq. (5.6) is tensor (integer spin) or spinor (half-integer spin) respectively. This is the spin–statistics theorem, first obtained for free fields by Fierz [25] and Pauli [26]; the general proof [27] followed much later.
- (b) The product *PCT* of parity, charge conjugation and time reversal is always a symmetry of a local quantum field theory. The *PCT* theorem was first stated in this general form by Pauli [28].

⁴For different components $i \neq j$, including the case of a field and its adjoint or a bosonic and a fermionic component, the “wrong” symmetry may occur. However, such a theory must then have the special property that new field components with the “right” symmetry can be constructed from the old fields via a “Klein transformation” [20].

5.2 Free Fields

The arguments leading to the formula (3.75) for the free electromagnetic four-potential can be readily carried over to other types of field. Indeed, the analysis is simpler for spin-0 and spin- $\frac{1}{2}$ fields because there is no gauge-fixing.

Instead of Maxwell's theory, we consider a general Lagrangian (density) with field dependence

$$\mathcal{L} = \mathcal{L}(\phi_1, \partial\phi_1, \dots, \phi_j, \partial\phi_j, \dots) \quad (5.10)$$

For complex components, the field and its conjugate are treated as separate variables in Eq. (5.10). According to classical field theory, this Lagrangian corresponds to the equations of motion

$$\frac{\partial\mathcal{L}}{\partial\phi_j} = \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\phi_j} \quad (5.11)$$

and the Hamiltonian (density)

$$\mathcal{H} = \sum_j \frac{\partial\mathcal{L}}{\partial\partial_0\phi_j} \partial_0\phi_j - \mathcal{L} \quad (5.12)$$

Free fields satisfy linear equations such as the equation for a free scalar field ϕ ,

$$(\partial^2 + m^2)\phi = 0 \quad (5.13)$$

so the Lagrangian is a quadratic form, e.g.

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \quad \text{real } \phi \quad (5.14)$$

or

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\phi^\dagger\phi, \quad \text{complex } \phi \quad (5.15)$$

Evidently, there are no polarization factors in the analogue of Eq. (3.75) for a spin-0 field:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2k_0} \{a(\mathbf{k})e^{-ik\cdot x} + b^\dagger(\mathbf{k})e^{ik\cdot x}\}, \quad k_0 = \sqrt{|\mathbf{k}|^2 + m^2} \quad (5.16)$$

If ϕ is real, the creation operator $b^\dagger(\mathbf{k})$ is just $a^\dagger(\mathbf{k})$, as in Eq. (3.75). Otherwise, $b^\dagger(\mathbf{k})$ creates particles which are distinct from those created by $a^\dagger(\mathbf{k})$; indeed they are anti-particles having the same mass but opposite charge. Thus complex fields are required when particles and their anti-particles are distinct — unlike photons, which are their own anti-particles.

The normalizations of the Lagrangians (5.14) and (5.15) are fixed by convention. Then an argument similar to that leading to Eqs. (3.55) and (3.75) determines constant factors in Eq. (5.16) by requiring $\int d^3x : \mathcal{H}(x) :$ to be the energy operator, with normal ordering as in Eq. (3.53).

The commutators of the free scalar field ϕ involve the massive version of the invariant function $\Delta^+(x)$ discussed in Section 4.1,

$$\Delta^+(x; m) = -i \lim_{\eta \rightarrow 0} \int \frac{d^3k}{(2\pi)^3 2k_0} e^{-ik\cdot(x-i\eta)}, \quad k_0 = \sqrt{|\mathbf{k}|^2 + m^2} \quad (5.17)$$

where the symbol \rightarrow indicates that the limit is taken through positive time-like values of η . If ϕ is real, direct calculation using Eq. (3.73) yields

$$[\phi(x), \phi(y)] = i\Delta(x - y; m) \quad (5.18)$$

where the massive Pauli-Jordan function [8]

$$\begin{aligned} \Delta(x; m) &= \Delta^+(x; m) - \Delta^+(-x; m) \\ &= -\frac{1}{2\pi}\varepsilon(x_0) \left\{ \delta(x^2) - \frac{m}{2\sqrt{x^2}}\theta(x^2)J_1\left(m\sqrt{x^2}\right) \right\} \end{aligned} \quad (5.19)$$

contains a Bessel function $J_\nu(z)$ of order $\nu = 1$. When ϕ is complex, $a(\mathbf{k})$ and $b(\mathbf{k})$ and their conjugates separately obey Eq. (3.73) but otherwise commute, so we find

$$[\phi(x), \phi^\dagger(y)] = i\Delta(x - y; m) \ , \quad [\phi(x), \phi(y)] = 0 \quad (5.20)$$

It is evident from the explicit expression (5.19) that these commutators exhibit microcausality. Notice that $\Delta^+(x; m)$ and hence $\Delta(x; m)$ are annihilated by the massive wave operator $(\partial^2 + m^2)$.

The free spin- $\frac{1}{2}$ field $\psi(x)$ satisfies the Dirac equation⁵

$$(i\cancel{\partial} - m)\psi(x) = 0 \quad (5.21)$$

where Feynman's "slash notation"

$$\cancel{A} = \gamma^\mu A_\mu \quad (5.22)$$

is used for the first-order differential operator $\cancel{\partial} = \gamma^\mu \partial_\mu$, and the 4×4 matrices γ^μ satisfy the Clifford algebra

$$[\gamma_\mu, \gamma_\nu]_+ = 2g_{\mu\nu} \quad (5.23)$$

Note that the matrix operator in (5.21) is really $i\cancel{\partial} - mI$, where I is the unit 4×4 matrix. Similarly, the right-hand side of Eq. (5.23) is $2g_{\mu\nu}I$.

Usually, ψ is associated with particles such as electrons which are distinct from their anti-particles (positrons), so its Fourier decomposition is of the form

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3 2p_0} \{ c_s(\mathbf{p})u_s(\mathbf{p})e^{-ip \cdot x} + d_s^\dagger(\mathbf{p})v_s(\mathbf{p})e^{ip \cdot x} \} \ , \quad p_0 = \sqrt{|\mathbf{p}|^2 + m^2} \quad (5.24)$$

where the operators $c_s(\mathbf{p})$ and $d_s(\mathbf{p})$ annihilate (say) electrons and positrons of spin projection s (up or down in the rest frame), and the corresponding four-spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ satisfy momentum-space Dirac equations

$$(\cancel{p} - m)u_s(\mathbf{p}) = 0 \ , \quad (\cancel{p} + m)v_s(\mathbf{p}) = 0 \quad (5.25)$$

In terms of inner products taken between spinors and their adjoints

$$\overline{\text{spinor}} = \{\text{spinor}\}^\dagger \gamma^0 \quad (5.26)$$

⁵Relativistic quantum mechanics is covered in the first volume of Bjorken and Drell [29] and in Chapter 2 of Itzykson and Zuber [9].

the momentum-space spinors $u_s(\mathbf{p}), v_s(\mathbf{p})$ are mutually orthogonal

$$\bar{u}_s(\mathbf{p})u_{s'}(\mathbf{p}) = 2m\delta_{ss'} = -\bar{v}_s(\mathbf{p})v_{s'}(\mathbf{p}) , \quad \bar{u}_s(\mathbf{p})v_{s'}(\mathbf{p}) = 0 \quad (5.27)$$

in a convenient normalization. Therefore they satisfy the following completeness relation in four-spinor space:

$$\sum_s \{u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) - v_s(\mathbf{p})\bar{v}_s(\mathbf{p})\} = 2m \quad (5.28)$$

To separate Eq. (5.28) into projectors for positive and negative energy solutions, left-multiply by the matrices $\not{p} \pm m$ and apply Eq. (5.25):

$$\sum_s u_s(\mathbf{p})\bar{u}_s(\mathbf{p}) = \not{p} + m , \quad \sum_s v_s(\mathbf{p})\bar{v}_s(\mathbf{p}) = \not{p} - m \quad (5.29)$$

Remarks:

(a) If ψ is massless ($m = 0$), the label s refers to helicity, and the conditions

$$u_s^\dagger(\mathbf{p})u_{s'}(\mathbf{p}) = 2p_0\delta_{ss'} = v_s^\dagger(\mathbf{p})v_{s'}(\mathbf{p}) \quad (5.30)$$

can be used instead of Eq. (5.27). Experimental evidence is consistent with neutrinos and anti-neutrinos having no mass.

(b) Theoretically it is possible for a fermion to be its own anti-fermion. Then the operators $c_s(\mathbf{p})$ and $d_s(\mathbf{p})$ are identical and ψ is called a *Majorana field*.

Given the normalizations used in Eqs. (5.24) and (5.27), the Lagrangian for Eq. (5.21) can be written

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi \quad (5.31)$$

An alternative, which treats ψ and $\bar{\psi}$ symmetrically, is

$$\tilde{\mathcal{L}} = \bar{\psi}(\frac{i}{2}\overleftrightarrow{\not{\partial}} - m)\psi \quad (5.32)$$

where $\overleftrightarrow{\partial}_\mu$ is defined by the formula

$$A\overleftrightarrow{\partial}_\mu B \equiv A(\partial_\mu B) - (\partial_\mu A)B \quad (5.33)$$

The difference between Eqs. (5.31) and (5.32) is a four-divergence $\frac{1}{2}\partial_\mu\{\bar{\psi}\gamma^\mu\psi\}$ which does not contribute to the equation of motion (5.11). Both Lagrangians contain an extra factor $\frac{1}{2}$ if the field is Majorana, as in the Lagrangian (5.14) for a real scalar field.

The operators $c_s(\mathbf{p})$ and $d_s(\mathbf{p})$ destroy fermions, so anti-commutator relations [30] are necessary:

$$\begin{aligned} [c_s(\mathbf{p}), c_{s'}^\dagger(\mathbf{q})]_+ &= [d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{q})]_+ = 2p_0\delta_{ss'}(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}) , \\ [c_s(\mathbf{p}), c_{s'}(\mathbf{q})]_+ &= [c_s(\mathbf{p}), d_{s'}(\mathbf{q})]_+ = [c_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{q})]_+ = [d_s(\mathbf{p}), d_{s'}(\mathbf{q})]_+ = 0 \end{aligned} \quad (5.34)$$

Indeed, when Eq. (5.24) is substituted into the Hamiltonian derived from \mathcal{L} or $\tilde{\mathcal{L}}$, normal ordering must include a minus sign for each fermion interchange

$$: c_s(\mathbf{p})c_{s'}^\dagger(\mathbf{q}) := - : c_{s'}^\dagger(\mathbf{q})c_s(\mathbf{p}) : , \quad : d_s(\mathbf{p})d_{s'}^\dagger(\mathbf{q}) := - : d_{s'}^\dagger(\mathbf{q})d_s(\mathbf{p}) : \quad (5.35)$$

if the desired result

$$\int d^3x : \mathcal{H} : = \int \frac{d^3p}{(2\pi)^3 2p_0} \{c_s^\dagger(\mathbf{p})c_s(\mathbf{p}) + d_s^\dagger(\mathbf{p})d_s(\mathbf{p})\} p_0 \quad (5.36)$$

is to be obtained. Similarly, the field anti-commutator can be calculated from Eqs. (5.24), (5.29) and (5.34) to obtain a result

$$[\psi_\sigma(x), \bar{\psi}_\tau(y)]_+ = (i\cancel{\partial} + m)_{\sigma\tau} i\Delta(x - y; m) \quad (5.37)$$

consistent with microcausality. As one might expect from the spin-statistics theorem, consistent results for ψ are not possible if one tries to use commutators throughout.

5.3 Canonical Methods

We must now consider how to quantize interacting field theories such that explicit calculations can be performed. The most popular procedures are:

- (a) Canonical quantization, pioneered by Heisenberg and Pauli [31];
- (b) Functional methods, particularly Feynman's path-integral approach [32].

Generally, calculations are possible only if one expands in the strength of the interaction, in successive orders of perturbation theory.

An example of an interacting theory is the scalar-field Lagrangian (5.14) with an extra term $-\lambda\phi^4$,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 \quad (5.38)$$

where the real coupling constant λ fixes the strength of the interaction. If so desired, another interaction could be included by adding the free-fermion Lagrangian (5.31) and a ‘‘scalar Yukawa coupling’’ $G\bar{\psi}\psi\phi$ to Eq. (5.38),

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \lambda\phi^4 + \bar{\psi}(i\cancel{\mathcal{D}} - m + G\phi)\psi \quad (5.39)$$

where G is (in this case) also real. Both λ and G become expansion parameters for the perturbative expansion.

A more practical example is quantum electrodynamics⁶ (QED), obtained from (5.31) by making the derivative covariant

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - ieA_\mu \quad (5.40)$$

and adding the Lagrangian $-\frac{1}{4}F^2$ for pure Maxwell theory:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\cancel{\mathcal{D}} - m)\psi \quad (5.41)$$

⁶Both QED and terms from (5.39) are contained in the electroweak sector of the Standard Model [5, 6, 9, 10] which is now the basis for particle physics phenomenology [33].

Here $-e$ is the “bare” electronic charge. The measured or “renormalized” charge -1.6×10^{-19} Coulomb corresponds to $-e$ plus perturbative corrections due to self-interactions of the electron.

Canonical theory treats fields $\phi_j(x)$ as dynamical variables evaluated at an *instant* of time. Three-dimensional smearing

$$\phi_{j,t}[g] = \int d^3x g(\mathbf{x}) \phi_j(t, \mathbf{x}) \quad (5.42)$$

is assumed to be adequate — time t is relegated to the status of a continuous label. Distributions with this property are said to be “sharp” in time.

The procedure resembles that of ordinary quantum mechanics, where independent dynamical variables $q_m(t)$ and their conjugate momenta $p_m(t)$ are required to satisfy

$$[q_m(t), p_n(t)] = i\delta_{mn} \ , \ [q_m(t), q_n(t)] = 0 = [p_m(t), p_n(t)] \quad (5.43)$$

The only difference is that the Kronecker delta has to be generalized to include a three-dimensional delta function. So fields $\phi_j(t, \mathbf{x})$ and their “generalized momenta”

$$\pi_j(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \phi_j(t, \mathbf{x})} \quad (5.44)$$

are postulated to obey the equal-time relations⁷ [31]

$$[\phi_j(t, \mathbf{x}), \pi_k(t, \mathbf{y})]_{\mp} = i\delta_{jk}\delta^3(\mathbf{x} - \mathbf{y}) \quad (5.45)$$

$$[\phi_j(t, \mathbf{x}), \phi_k(t, \mathbf{y})]_{\mp} = 0 = [\pi_j(t, \mathbf{x}), \pi_k(t, \mathbf{y})]_{\mp} \quad (5.46)$$

As usual, anticommutators are required if both ϕ_j and ϕ_k are fermionic.

The canonical hypotheses (5.45) and (5.46) can be tested [31] by taking the equal-time limit of known free-field (anti-)commutators such as (5.20) and (5.37). These involve the invariant function $\Delta(x; m)$ of Eq. (5.19), whose singular part at short distances $x_\mu \rightarrow 0$ is given entirely by the mass-independent function $\Delta(x)$ of Eq. (4.15):

$$\Delta(x; m) = -\frac{1}{2\pi}\varepsilon(x_0)\delta(x^2) + \frac{m^2}{4\pi}\varepsilon(x_0)\theta(x^2)\{1 + O(m^2x^2)\} \ , \ x_\mu \rightarrow 0 \quad (5.47)$$

Let us apply the smearing operation $\int d^3x g(\mathbf{x}) \dots$ to this expansion, and consider the region $t \sim 0$. The mass-dependent part is finite and confined to the region $|\mathbf{x}| \leq |t|$, so its asymptotic contribution is

$$\frac{m^2}{4\pi}\varepsilon(x_0) \int_0^{|t|} d|\mathbf{x}| |\mathbf{x}|^2 \int d\Omega g(\mathbf{x}) \sim \frac{1}{3}m^2 g(0) t^3 \ , \ t \rightarrow 0 \quad (5.48)$$

where $d\Omega$ is the element of solid angle in the direction $\hat{\mathbf{x}}$. Therefore, we have

$$\begin{aligned} \int d^3x \Delta(t, \mathbf{x}; m)g(\mathbf{x}) &= \int d^3x \Delta(t, \mathbf{x})g(\mathbf{x}) + O(t^3) \\ &= -\frac{t}{4\pi} \int d\Omega g(|t|\hat{\mathbf{x}}) + O(t^3) \end{aligned} \quad (5.49)$$

⁷This assumes that there are no constraints on the dynamical variables. The general theory for constrained systems was developed by Dirac [34, 9]. In quantum electrodynamics and other gauge theories, this is associated with the need to fix a gauge.

where the last line follows from Eq. (4.22) for $\Delta(x)$. Then we can substitute

$$g(|t|\hat{\mathbf{x}}) = \{1 + |t|\hat{\mathbf{x}} \cdot \nabla + O(t^2)\} g(0) \quad (5.50)$$

and note that the angular integral $\int d\Omega \hat{\mathbf{x}}$ vanishes:

$$\int d^3x \Delta(t, \mathbf{x}; m) g(\mathbf{x}) = -t g(0) + O(t^3) \quad (5.51)$$

The relevant equal-time relations can be read from Eq. (5.51). There are no $O(t^0)$ or $O(t^2)$ terms,

$$\lim_{t \rightarrow 0} \Delta(t, \mathbf{x}; m) = 0 = \lim_{t \rightarrow 0} \partial_0^2 \Delta(t, \mathbf{x}; m) \quad (5.52)$$

but there is a term $O(t)$ which gives rise to the key result [31]:

$$\lim_{t \rightarrow 0} \partial_0 \Delta(t, \mathbf{x}; m) = -\delta^3(\mathbf{x}) \quad (5.53)$$

Then it is evident that the postulates (5.45) and (5.46) work for free fields. For example:

- (a) The Lagrangian (5.15) for a free complex scalar field ϕ requires $\partial_0 \phi^\dagger$ to be conjugate to ϕ . From the commutators (5.20) and Eqs. (5.52) and (5.53), we find

$$[\phi(t, \mathbf{x}), \partial_0 \phi^\dagger(t, \mathbf{y})] = i \delta^3(\mathbf{x} - \mathbf{y}) \quad (5.54)$$

with other possibilities $[\phi, \phi]$, $[\phi, \phi^\dagger]$, $[\partial_0 \phi, \partial_0 \phi^\dagger]$, and $[\partial_0 \phi^\dagger, \partial_0 \phi^\dagger]$ vanishing at equal times.

- (b) At equal times, the free-fermion anti-commutator (5.37) reduces to

$$[\psi_\sigma(t, \mathbf{x}), \bar{\psi}_\tau(t, \mathbf{y})]_+ = (\gamma^0)_{\sigma\tau} \delta^3(\mathbf{x} - \mathbf{y}) \quad (5.55)$$

which is consistent with $i\psi^\dagger$ being conjugate to ψ .

In textbooks, Eq. (5.53) is often obtained by combining Eq. (5.17) for $\Delta^+(x; m)$ with the first equality in Eq. (5.19) and neglecting the limiting procedure $\eta_\mu \rightarrow 0$:

$$\lim_{t \rightarrow 0} \partial_0 \Delta(t, \mathbf{x}; m) = -i \int \frac{d^3k}{(2\pi)^3 2k_0} \partial_0 \{e^{-ik \cdot x} - e^{ik \cdot x}\}_{t=0} \quad (5.56)$$

However this obscures the role of short-distance behaviour in determining the result at equal times.

The derivation leading to Eqs. (5.52) and (5.53) illustrates a general rule [35] governing equal-time limits. Non-zero contributions to *any* causal (anti-)commutator

$$[\phi_j(x_0, \mathbf{x}), \phi_k(y_0, \mathbf{y})]_\mp \quad (5.57)$$

are restricted to the region

$$|\mathbf{x} - \mathbf{y}| \leq |x_0 - y_0| \quad (5.58)$$

Therefore all equal-time commutators of the operators ϕ_j and of their derivatives are *entirely* determined by the asymptotic behaviour of (5.57) at short distances

$$(x - y)_\mu \sim 0 \tag{5.59}$$

This observation is critical in determining the status of the canonical commutation relations (5.45) and (5.46). The aim of these postulates is to state general quantization conditions for *interacting* theories. If the postulates are *really* valid, the short-distance behaviour of the theory should not depend on the interaction: it should be the *same* as for free fields. It turns out that this requirement *is* satisfied in the lowest-order “tree” approximation (sec. 6.3 below), but generally *not* in higher orders where self-interactions occur.

This contradiction of the canonical postulates is caused by *renormalization*, a “patch-up” procedure used to absorb infinite self-interaction ambiguities into physical constants such as charges and masses. Despite its *ad hoc* appearance, renormalization produces a theory which satisfies general physical requirements such as causality, unitarity of the S -matrix, and above all, agreement with experiment. Originally it was invented as the final step in the evolution of quantum electrodynamics from Dirac’s 1927 theory [3], with matter represented by non-relativistic wave functions, to the complete relativistic field theory of Schwinger, Feynman, Tomonaga and Dyson, in which all divergences are eliminated from perturbative calculations. (The history of these developments can be traced in the well-known collection of classic papers selected by Schwinger [36].) Space and time do not allow me to cover renormalization in any detail, so readers are referred to general textbooks [4–10] and the book by Collins [37]. However the general idea is as follows.

Perturbative corrections to amplitudes such as the strength of an interaction depend on the values of incoming and outgoing momenta. Therefore, if we are to identify a coupling “constant” as a measurable parameter, we have to specify a point in momentum space to which it refers. This introduces into the theory a new scale μ with dimensions of mass: amplitudes depend on μ as well as momenta and masses.

At short distances, the dependence on masses disappears, as in Eq. (5.47), but the μ -dependence survives. This is because infinite ambiguities due to self interactions are ultra-violet problems which persist even in the massless case. Generally, each new order of perturbation theory introduces an extra logarithmic factor $\ln(\mu^2 x^2)$ at short distances. At a given perturbative order, the result for scalar fields is

$$[\phi_j(x), \pi_k(y)] \sim C \partial_0 \text{Im} \frac{\ln^p \{ \mu^2 (-(x - y)^2 + i\epsilon(x_0 - y_0)) \}}{(x - y)^2 - i\epsilon(x_0 - y_0)}, \quad x_\mu - y_\mu \rightarrow 0 \tag{5.60}$$

where p is a positive integer, C is a calculable constant, and Im denotes the imaginary part. In the corresponding anti-commutator for spin- $\frac{1}{2}$ fields, ∂_0 is replaced by $\not{\partial}\gamma_0$.

As explained above, the short-distance result (5.60) enables us to *calculate* what happens at equal times $x_0 \sim y_0$:

$$[\phi_j(x_0, \mathbf{x}), \pi_k(y_0, \mathbf{y})]_{\mp} \sim 2^{p+1} \pi^2 C \ln^p(\mu|x_0 - y_0|) \delta^3(\mathbf{x} - \mathbf{y}) \tag{5.61}$$

Evidently the result is an asymptotic series in powers of $\ln(\mu|x_0 - y_0|)$. It does *not* converge in the limit $x_0 - y_0 \rightarrow 0$. Therefore the canonical postulates (5.45) and (5.46) *must be abandoned*.

This crucial point was first recognised by axiomatic theorists, as reported on p. 101 of Streater and Wightman [20]. It entered the mainstream literature twenty-five years ago with the appearance of Wilson’s well-known work on operator products at short distances [35, 37]. Nevertheless, textbooks are still appearing with the reverential label “fundamental” attached to the canonical commutators (5.45) and (5.46). Alternatively, authors try to incorporate renormalization into an equal-time framework by introducing a cutoff Λ used at intermediate stages of calculations to control infinities and writing

$$[\phi_j(t, \mathbf{x}), \pi_k(t, \mathbf{y})]_{\mp} \stackrel{?}{=} iZ(\Lambda)\delta_{jk}\delta^3(\mathbf{x} - \mathbf{y}) , \quad \Lambda \sim \infty \quad (5.62)$$

where the constant $Z(\Lambda)$ is a power series in $\ln(\Lambda/\mu)$. This is at best misleading; the fact is that, in perturbation theory, equal-time limits diverge as powers of $\ln(\mu|x_0 - y_0|)$.

The result (5.61) shows that the operators $\phi_j(x)$ and $\pi_j(x)$ are not sharp in time. Each operator smeared over a space-time region, no matter how small, is partly time-like relative to its immediate neighbours, so it can be causally affected by them. Consequently, it cannot be assumed that the fixed- t dynamical variables $\{\phi_j(t, \mathbf{x})\}$ are entirely independent of each other, even though microcausality is obeyed exactly. This has thwarted attempts to quantize fields by analogy with the standard prescription (5.43) for quantum mechanics. It is not clear that such a prescription is either possible or necessary, but it would be an interesting development if someone could provide a suitable construction. In the early eighties, Symanzik [38] considered formulating quantum field theory in the Schrödinger picture, but his line has not been pursued since then.

Other canonical procedures are also vitiated by renormalization. For example, normal ordering of composite operators such as the Hamiltonians (3.53) and (5.36) is no longer adequate — in higher orders, additional operator subtractions become necessary. The classical Noether construction for the current j^μ associated with a transformation $\delta\phi_j$ on the fields ϕ_j is particularly vulnerable, because it requires us to multiply two operators at the same point:

$$j^\mu = \sum_j \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_j} \delta\phi_j \quad (5.63)$$

After renormalization, there is no guarantee that the divergence $\partial_\mu j^\mu$ will be given by $\delta\mathcal{L}$. The axial-vector anomaly [39] is the most famous example of a Noether construction breaking down.

Whenever renormalization produces a result at variance with canonical expectations, the effect is said to be “anomalous”. Anomalies have occupied a central place in field-theoretic research during the past twenty-five years, with direct applications to phenomenology, especially in quantum chromodynamics [40, 41], the

gauge theory of quarks and gluons. Consequently, I have taken some trouble to emphasize the incompatibility of renormalization and canonical theory.

I should conclude this criticism of canonical procedures by noting that similar problems are encountered with functional methods. As we shall see, the latter rely on the assumption that unsmeared fields which are space-like separated or Euclidean are dynamically independent, so that they can be diagonalized simultaneously.

5.4 Continuation to Euclidean Space

Local quantum field theories have one other general property worth noting: it is always possible to continue amplitudes analytically in x_μ away from Minkowski space-time with invariant interval

$$x^\mu x_\mu = x_0^2 - \mathbf{x}^2 \quad (5.64)$$

to Euclidean four-space:

$$x_\mu x_\mu = \mathbf{x}^2 + x_4^2 \quad (5.65)$$

In any quantum theory, there must be a state of lowest energy, the “ground state”. In a quantum field theory, the ground state is known as the vacuum $|\text{vac}\rangle$. Generally, it lacks any particle structure or other physical characteristics attached to a particular frame of reference. Thus the property which distinguishes $|\text{vac}\rangle$ from other state vectors is its Poincaré invariance:

$$P_\mu |\text{vac}\rangle = 0, \quad U_\Lambda |\text{vac}\rangle = 0 \quad (5.66)$$

An example of such a state is the no-photon state $|0\rangle$ in free-photon theory. The requirement that it have zero energy led to normal ordering for the photon Hamiltonian in Eqs. (3.52) and (3.53). Free-field vacua satisfy special conditions of the form

$$a_{\mathbf{k}\lambda} |0\rangle = 0 \quad (5.67)$$

Generally Eq. (5.67) is not applicable if there are interactions, but Eq. (5.66) retains its validity, irrespective of the properties of other states in the theory.

Given $|\text{vac}\rangle$, we can construct vacuum expectation values

$$\mathcal{W} = \langle \text{vac} | \phi_1(x_1) \phi_2(x_2) \dots \phi_n(x_n) | \text{vac} \rangle \quad (5.68)$$

known as “unordered” or “Wightman” functions. According to the translation property (5.4), the dependence on the space-time points $(x_\mu)_j$ can be extracted in the following way:

$$\phi_j(x_j) = e^{iP \cdot x_j} \phi_j(0) e^{-iP \cdot x_j} \quad (5.69)$$

Eq. (5.66) requires the vacuum state to be translation invariant,

$$e^{-iP \cdot x} |\text{vac}\rangle = |\text{vac}\rangle \quad (5.70)$$

so \mathcal{W} depends on coordinate differences only:

$$\mathcal{W} = \mathcal{W}(x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n) \quad (5.71)$$

A simple example is the two-point function for the scalar field (5.16):

$$\mathcal{W}(x-y) = \langle 0|\phi(x)\phi^\dagger(y)|0\rangle = i\Delta^+(x;m) \quad (5.72)$$

According to the definition (5.17) of Δ^+ , the result is a boundary value

$$\mathcal{W}(x-y) = \lim_{\eta \rightarrow 0} i\Delta^+(z;m) \quad (5.73)$$

where

$$z_\mu = x_\mu - y_\mu - i\eta_\mu \quad (5.74)$$

is a complex four-vector and η_μ lies within the forward light cone. The case $m=0$ is discussed in detail in Eqs. (4.9) to (4.12).

This connection with complex functions is a general property of vacuum expectation values [20, 42]. It is derived from the fact that all eigenvalues p_μ of the four-momentum operator P_μ must lie on or within the forward light cone:

$$p^2 \geq 0, \quad p_0 \geq 0 \quad (5.75)$$

To see this, insert sets of complete states $|I\rangle$ between the operators ϕ_i in (5.68),

$$\mathcal{W} = \sum_{I_1 \dots I_{n-1}} \langle \text{vac}|\phi_1(x_1)|I_1\rangle \langle I_1|\phi_2(x_2)|I_2\rangle \dots \langle I_{n-1}|\phi_n(x_n)|\text{vac}\rangle \quad (5.76)$$

and use the translation properties (5.69) and (5.70),

$$\begin{aligned} \mathcal{W} = & \sum_{I_1 \dots I_{n-1}} e^{-ip_1 \cdot (x_1 - x_2)} e^{-ip_2 \cdot (x_2 - x_3)} \dots e^{-ip_{n-1} \cdot (x_{n-1} - x_n)} \\ & \times \langle \text{vac}|\phi_1(0)|I_1\rangle \langle I_1|\phi_2(0)|I_2\rangle \dots \langle I_{n-1}|\phi_n(0)|\text{vac}\rangle \end{aligned} \quad (5.77)$$

where $(p_k)_\mu$ is the appropriate four-momentum eigenvalue:

$$P_\mu |I_k\rangle = (p_k)_\mu |I_k\rangle \quad (5.78)$$

Each summation \sum_{I_k} includes an integral over $(p_k)_\mu$ which has to be regulated at large momenta, as in Eq. (4.10). Each coordinate $(x_k)_\mu$ is complexified

$$x_k \longrightarrow z_k \quad (5.79)$$

such that successive differences

$$z_k - z_{k+1} = x_k - x_{k+1} - i\eta_k, \quad k = 1, 2, \dots, n-1 \quad (5.80)$$

acquire imaginary parts $-\eta_k$ with each real four-vector $(\eta_k)_\mu$ restricted to lie within the forward light cone. This ensures that each oscillatory exponential in (5.77) acquires a damping factor at large p_k :

$$e^{-ip_k \cdot (x_k - x_{k+1})} \longrightarrow e^{-ip_k \cdot (x_k - x_{k+1})} e^{-p_k \cdot \eta_k}, \quad p_k \cdot \eta_k > 0 \quad (5.81)$$

Then the unordered function (5.71) is obtained as a boundary value

$$\mathcal{W}(x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n) = \lim_{\eta_1 \dots \eta_{n-1} \rightarrow 0} W(z_1, z_2, \dots, z_n) \quad (5.82)$$

of a function W of several complex variables:

$$W(z_1, z_2, \dots, z_n) = \sum_{I_1 \dots I_{n-1}} \langle \text{vac} | \phi_1(0) | I_1 \rangle \langle I_1 | \phi_2(0) | I_2 \rangle \dots \langle I_{n-1} | \phi_n(0) | \text{vac} \rangle e^{-i \sum_k p_k \cdot (z_k - z_{k+1})} \quad (5.83)$$

Note that vacuum expectation values of field products $\Pi_j \phi_j(x_j)$ written in various orders correspond to different boundary values of the same complex function (5.83). To obtain

$$\langle \text{vac} | \phi_{j_1}(x_{j_1}) \phi_{j_2}(x_{j_2}) \dots \phi_{j_n}(x_{j_n}) | \text{vac} \rangle \quad (5.84)$$

simply replace Eq. (5.80) by

$$z_{j_k} - z_{j_{k+1}} = x_{j_k} - x_{j_{k+1}} - i\eta_k, \quad \eta_k^2 > 0, \quad \eta_k^0 > 0 \quad (5.85)$$

and take the limit $\eta_1 \dots \eta_n \rightarrow 0$ of $W(z_1 \dots z_n)$.

If neighbouring operators in Eq. (5.84) are space-like separated and so commute (or anti-commute), the two boundary values corresponding to their interchange coincide. The absence of any discontinuity indicates that W is analytic in the corresponding z variables. Indeed, the space-like region defined by

$$(x_j - x_k)^2 < 0, \quad \text{all } j, k \quad (5.86)$$

lies within the domain of analyticity \mathcal{D} of W as a function of z_1, z_2, \dots, z_n .

The space-like region (5.86) can be extended to a special subspace of \mathcal{D} called the *Euclidean region*. It consists of non-coincident points $\{z_1, \dots, z_n\}$ for which each z_k has real space components and pure imaginary time components:

$$z_k \longrightarrow z_k^E = (-ix_4, \mathbf{x})_k, \quad z_j^E \neq z_k^E \quad (5.87)$$

The real variables \mathbf{x}_k and $(x_k)_4$ can be treated as the components of a four dimensional Euclidean vector $(x_k)_\mu$, $\mu = 1, 2, 3, 4$:

$$\begin{aligned} \mathbf{x}_k &= (\mathbf{x}, x_4)_k \\ (x_k)_\mu \cdot (x_k)_\mu &= \mathbf{x}_k^2 + (x_k)_4^2 = -z_k^E \cdot z_k^E \end{aligned} \quad (5.88)$$

Amplitudes W evaluated at these points are called Euclidean Green's functions, or Schwinger functions:

$$\mathcal{G}(x_1, \dots, x_n) = W(z_1^E, \dots, z_n^E) \quad (5.89)$$

These functions can be regarded as amplitudes of a Euclidean version of the theory in which Euclidean field operators $\Phi_j(x_j)$ act on a state vector space distinct from but analogous to that for the Minkowskian theory. In particular, there is a Euclidean "vacuum" state $|\text{vac}\rangle$ distinguished by its invariance under Euclidean translations and $SO(4)$ rotations. Euclidean Green's functions are given by vacuum expectation values

$$\mathcal{G}(x_1, \dots, x_n) = \langle \text{vac} | \Phi_1(x_1) \dots \Phi_n(x_n) | \text{vac} \rangle \quad (5.90)$$

which depend on Euclidean coordinate differences $x_1 - x_2, \dots, x_{n-1} - x_n$. Because of analyticity, there are no discontinuities associated with different operator orderings, so we have

$$[\Phi_j(x_j), \Phi_k(x_k)]_{\mp} = 0 \quad (5.91)$$

throughout the Euclidean region. It is this feature which makes Euclidean operators easier to handle than their Minkowskian counterparts.

The commutativity property (5.91) does not mean that the Euclidean theory is classical. Its quantum nature becomes evident when Euclidean Green's functions \mathcal{G} are extended to include coincident points $x_j = x_k$, which lie on the boundary of the analyticity domain \mathcal{D} .

For example, consider the free two-point function (5.72) with mass $m = 0$ continued to the Euclidean region. From Eq. (4.11), we find

$$\mathcal{G}(x - y) = \langle \text{vac} | \Phi(x) \Phi^\dagger(y) | \text{vac} \rangle = \frac{1}{4\pi^2(x - y)^2} \quad (5.92)$$

The equation of motion satisfied by the free Euclidean field $\Phi(x)$ is

$$\partial^2 \Phi \equiv \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial x_\mu} \Phi = 0 \quad (5.93)$$

but for the amplitude (5.92), we find

$$\partial^2 \mathcal{G}(x - y) = -\delta^4(x - y) \quad (5.94)$$

instead. That is why the term ‘‘Green’s function’’ is appropriate for Euclidean amplitudes (5.89).

There is a close resemblance between Euclidean Green’s functions and time-ordered vacuum expectation values in the Minkowskian theory, which are also Green’s functions. In time-ordered products, each operator appears to the left of operators evaluated at earlier times and to the right of those at later times. Formally, this can be indicated by multiplying various orderings by step functions in time differences⁸ and summing all possibilities (with a minus sign for each fermion interchange). Thus for two operators, we have

$$T\{\phi_1(x_1)\phi_2(x_2)\} = \theta(x_1^0 - x_2^0)\phi_1(x_1)\phi_2(x_2) \pm \theta(x_2^0 - x_1^0)\phi_2(x_2)\phi_1(x_1) \quad (5.95)$$

where the minus sign applies if both ϕ_1 and ϕ_2 are fermionic. Then the interchangeability of Euclidean operators $\Phi_j(x_j)$ in (5.90) (due to Eq. (5.91)) is matched by a similar interchangeability of Minkowskian operators $\phi_j(x_j)$ in the corresponding time-ordered amplitudes

$$T\langle \text{vac} | \phi_1(x_1)\phi_2(x_2) \dots \phi_n(x_n) | \text{vac} \rangle \quad (5.96)$$

The time-ordered amplitude corresponding to (5.92)

$$T\langle 0 | \phi(x)\phi^\dagger(y) | 0 \rangle = \theta(x^0 - y^0)i\Delta^+(x - y) + \theta(y^0 - x^0)i\Delta^+(y - x) \quad (5.97)$$

⁸This works if the unordered functions are not too singular at short distances, i.e. as various subsets of coordinates become coincident; for example, two-point behaviour should be less singular than $(x - y)^{-4}$. Otherwise, a renormalized time ordering must be defined for each amplitude.

can be deduced from the result (4.12) for $\Delta^+(x)$:

$$\begin{aligned} T\langle 0|\phi(x)\phi^\dagger(y)|0\rangle &= -\frac{1}{4\pi^2} \left(\frac{\theta(x^0 - y^0)}{(x - y)^2 - i\epsilon(x^0 - y^0)} + \frac{\theta(y^0 - x^0)}{(y - x)^2 - i\epsilon(y^0 - x^0)} \right) \\ &= -\frac{1}{4\pi^2 ((x - y)^2 - i\epsilon)} \end{aligned} \quad (5.98)$$

This is the coordinate-space representation of the Feynman propagator for a massless scalar field. Note the simple substitution $x^2 \leftrightarrow -x^2$ which relates Eqs. (5.92) and (5.98).

The $-i\epsilon$ prescription for the singularity in (5.98) is responsible for it behaving as a Green's function:

$$\partial^2 T\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = -i\delta^4(x - y) \quad (5.99)$$

This should be compared with the result (4.13) for Δ^+ . Note that we could have deduced (5.99) by direct differentiation of the definition (5.95), taking into account the identity $\partial_\mu\theta(x^0) = g_{\mu 0}\delta(x^0)$ and the equation of motion $\partial^2\phi = 0$ and canonical commutator (5.54) for the free scalar field ϕ :

$$\partial_x^2 T\{\phi(x)\phi^\dagger(y)\} = [\partial_0\phi(x), \phi^\dagger(y)]\delta(x_0 - y_0) = -i\delta^4(x - y) \quad (5.100)$$

The relation between time-ordered and Euclidean amplitudes is equally direct in momentum space. In Minkowski space, the propagator (5.98) can be written

$$T\langle 0|\phi(x)\phi^\dagger(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 + i\epsilon} e^{-ik \cdot (x - y)} \quad (5.101)$$

and in the massive case, we get

$$\frac{i}{k^2 + i\epsilon} \longrightarrow \frac{i}{k^2 - m^2 + i\epsilon} \quad (5.102)$$

instead. The continuation to Euclidean space $k^2 \rightarrow -k^2$ permits the on-shell singularity at $k^2 = m^2$ to be avoided.

Perturbative amplitudes are essentially four-momentum integrals over products of the Feynman propagators (5.102), so it was natural that the idea to continue to Euclidean space should have arisen first in that context [43]. The formulation of Euclidean theory in terms of operators was first suggested by Schwinger [44], and developed as an alternative approach to axiomatic theory by Symanzik [45] for many years before it became fashionable.

The foregoing discussion dodges a good deal of very advanced analysis of interest to rigorous mathematical physicists. Devotees should consult the book by Simon [46].

6. Functional Methods

Feynman's method of "path" or "functional" integration [32] is based on an ingenious use of quantum superposition combined with some prescient remarks of Dirac [47] about the role of the action in quantum theory.

In ordinary quantum mechanics, as a system evolves in time t , it passes through a variable mixture of eigenstates of a given dynamical coordinate $Q(t)$. This corresponds to the fact that generally, the uncertainty $\Delta Q(t)$ is not zero; at each time t , a range of eigenvalues $\{q_n(t)\}$ of $Q(T)$ is involved. Therefore, the evolution of the system can be viewed as consisting of many sequences of hops from eigenvalue to eigenvalue

$$q_{n_0}(t_0) \rightarrow q_{n_1}(t_1) \dots \rightarrow q_{n_\ell}(t_\ell) \quad (6.1)$$

over a large number ℓ of small positive increments of time $t_k \rightarrow t_{k+1}$. Each sequence of hops can be considered to define a path $q(t)$. Then the full amplitude is given by an appropriate sum over all such paths, or path integral:

$$\text{amplitude} = \int [dq(t)] \{\text{integrand}\} \quad (6.2)$$

It turns out that the integrand is just $\exp i\{\text{Action}/\hbar\}$. A detailed account can be found in the book by Feynman and Hibbs [32].

In field theory, the path integral becomes a sum over functions φ of x_μ or its Euclidean cousin x_μ . We will consider the Euclidean case, where (for bosons at least) the commutativity property (5.91) allows *simultaneous diagonalization* of the set of Euclidean field operators $\Phi_j(x)$ labelled by *all* x_μ . This yields eigenvalues $\varphi_j(x)$ which are also labelled by x_μ :

$$\Phi_j(x)|\varphi_1, \dots, \varphi_n\rangle = \varphi_j(x)|\varphi_1, \dots, \varphi_n\rangle \quad (6.3)$$

It is these eigenvalues over which the functional summation is to be performed.

As in ordinary calculus, derivatives are easier than integrals, so we tackle functional differentiation first, cover path integration for bosons, and finally, deal with fermions, where a special treatment is necessary [48].

Functional integrals have played an important role in the development of gauge theories, and so they feature prominently in modern textbooks [9, 49–53].

6.1 Functional Differentiation

A functional F maps functions $\varphi(x)$ to numbers $F[\varphi]$:

$$\varphi(x) \xrightarrow{F} F[\varphi] \quad (6.4)$$

For example, for real functions φ , we can construct functionals such as

$$F[\varphi] = \int d^4x \varphi^2(x) \quad (6.5)$$

or

$$F[\varphi] = \int d^4x (\partial\varphi(x))^2 \quad (6.6)$$

which are quadratic in φ . Generalized functions correspond to functionals *linear* in the test function; the smeared operator (4.29) is an example.

The functional derivative of F is defined as the generalized function $\delta F/\delta\varphi(x)$ smeared over test functions $f(x)$

$$\int d^4x f(x) \frac{\delta F[\varphi]}{\delta\varphi(x)} = \lim_{\epsilon \rightarrow 0} \frac{F[\varphi + \epsilon f] - F[\varphi]}{\epsilon} \quad (6.7)$$

whenever the limit exists. Thus, for the example (6.5), we can calculate

$$\begin{aligned} F[\varphi + \epsilon f] &= \int d^4x \{ \varphi^2 + 2\epsilon f\varphi + \epsilon^2 f^2 \} \\ &= F[\varphi] + 2\epsilon \int d^4x f(x)\varphi(x) + O(\epsilon^2) \end{aligned} \quad (6.8)$$

and so conclude

$$\frac{\delta F[\varphi]}{\delta\varphi(x)} = 2\varphi(x) \quad (6.9)$$

Similarly, for the example (6.6), we have

$$F[\varphi + \epsilon f] - F[\varphi] = 2\epsilon \int d^4x (\partial f) \cdot (\partial\varphi) + O(\epsilon^2) = -2\epsilon \int d^4x f \partial^2\varphi + O(\epsilon^2) \quad (6.10)$$

and hence

$$\frac{\delta F[\varphi]}{\delta\varphi(x)} = -2\partial^2\varphi(x) \quad (6.11)$$

Sometimes it is stated that the functional derivative can be defined as follows:

$$\frac{\delta F[\varphi(y)]}{\delta\varphi(x)} \stackrel{?}{=} \lim_{\epsilon \rightarrow 0} \frac{F[\varphi(y) + \epsilon\delta^4(x-y)] - F[\varphi(y)]}{\epsilon} \quad (6.12)$$

The trouble with this idea can be seen from the example (6.5), for which we would have the ill-defined expression

$$F[\varphi(y) + \epsilon\delta^4(x-y)] \stackrel{?}{=} \int d^4y (\varphi(y) + \epsilon\delta^4(x-y))^2 \quad (6.13)$$

As in ordinary calculus, functional differentiation can be automated by combining basic results such as

$$\begin{aligned} \frac{\delta\varphi(y)}{\delta\varphi(x)} &= \delta^4(x-y) \\ \frac{\delta}{\delta\varphi(x)} \left\{ \frac{\partial}{\partial y_\mu} \varphi(y) \right\} &= \frac{\partial}{\partial y_\mu} \delta^4(x-y) \end{aligned} \quad (6.14)$$

with the product rule

$$\frac{\delta(FG)}{\delta\varphi} = \frac{\delta F}{\delta\varphi} G + F \frac{\delta G}{\delta\varphi} \quad (6.15)$$

and the chain rule

$$\frac{\delta}{\delta\psi(x)} = \int d^4y \frac{\delta\varphi(y)}{\delta\psi(x)} \frac{\delta}{\delta\varphi(y)} \quad (6.16)$$

Thus functional derivatives are as easy to calculate as ordinary derivatives.

Another rule worth noting is the formula for translations in function space:

$$F[\varphi + \xi] = \exp \left\{ \int d^4x \xi(x) \frac{\delta}{\delta\varphi(x)} \right\} F[\varphi] \quad (6.17)$$

This is a functional version of the Taylor expansion for ordinary functions f :

$$f(x + a) = e^{a \cdot \partial} f(x) \quad (6.18)$$

Functional derivatives occur naturally in field theories. The standard problem is that of determining the classical equation of motion by varying the action S and setting the result equal to zero. This can be done either directly, by treating S as a functional of φ , or via the Euler-Lagrange formalism, where the corresponding Lagrangian \mathcal{L} is treated as a function of φ and $\partial\varphi$, in the *ordinary* sense of the term “function”:

$$S[\varphi] = \int d^4y \mathcal{L}(\varphi(y), \partial\varphi(y)) \quad (6.19)$$

Note the use of square and curved brackets to distinguish dependence as a functional from that as a function. A relation between the two types of derivative is readily deduced:

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = \frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\partial_\mu\varphi} \quad (6.20)$$

At a minimum of S , we get the classical equation of motion either in Euler-Lagrange form (the right-hand side of (6.20) vanishing), or more simply, in functional form:

$$\frac{\delta S[\varphi]}{\delta\varphi(x)} = 0 \quad (6.21)$$

A formalism for the systematic use of functional derivatives in quantum field theory was developed by Schwinger [54].

6.2 Functional Integrals and the Action

Consider a single Euclidean Bose field Φ . Because of the commutativity property (5.91), we argue as in Eq. (6.3) that $\Phi = \Phi(x)$ can be treated as a collection of mutually commuting operators labelled by all Euclidean points x_μ . Therefore, we should be able to construct simultaneous eigenstates $|\varphi\rangle$ of this operator set:

$$\Phi(x)|\varphi\rangle = \varphi(x)|\varphi\rangle \quad (6.22)$$

The symbol φ inside $|\varphi\rangle$ represents the set of eigenvalues $\{\varphi(x), \text{ for all } x_\mu\}$. Presumably, these eigenstates span state-vector space, so there should be a completeness relation of the form

$$\int [d\varphi] |\varphi\rangle\langle\varphi| = 1 \quad (6.23)$$

where $\int [d\varphi]$ indicates that a sum over functions φ , or functional integral, is to be performed.

This statement of completeness specifies that we are dealing with a field theory. A less dense state structure, such as that corresponding to eigenstates of the

quantum coordinate $Q(t)$, would correspond to quantum mechanics. If a richer state structure is desired, eigenstates of string operators should be considered.

Since eigenstates are a functional $|\varphi\rangle$ of φ , their orthonormality must be expressed in terms of a delta functional:

$$\langle\varphi_1|\varphi_2\rangle = \delta[\varphi_1 - \varphi_2] \quad (6.24)$$

The delta functional should have the property

$$\int[d\varphi_1] \delta[\varphi_1 - \varphi_2] f[\varphi_1] = f[\varphi_2] \quad (6.25)$$

As a check, note that this is consistent with choosing a functional F

$$F[\varphi] = \langle\varphi|f\rangle$$

and applying completeness:

$$\int[d\varphi_1] \langle\varphi_2|\varphi_1\rangle \langle\varphi_1|F\rangle = \langle\varphi_2|F\rangle$$

Now consider the insertion of completeness relations around each operator in an n -point Euclidean Green's function:

$$\begin{aligned} &\langle\text{vac}|\Phi(x_1) \dots \Phi(x_n)|\text{vac}\rangle \\ &= \int[d\varphi_1] \dots \int[d\varphi_{n+1}] \langle\text{vac}|\varphi_1\rangle \langle\varphi_1|\Phi(x_1)|\varphi_2\rangle \dots \langle\varphi_n|\Phi(x_n)|\varphi_{n+1}\rangle \langle\varphi_{n+1}|\text{vac}\rangle \end{aligned} \quad (6.26)$$

Eqs. (6.22) and (6.23) imply

$$\langle\varphi_1|\Phi(x_1)|\varphi_2\rangle = \varphi_2(x_1) \langle\varphi_1|\varphi_2\rangle = \varphi_2(x_1) \delta[\varphi_1 - \varphi_2] \quad (6.27)$$

so all except one of the $[d\varphi_k]$ integrals is trivial:

$$\langle\text{vac}|\prod_{j=1}^n \{\Phi(x_j)\}|\text{vac}\rangle = \int[d\varphi] |\langle\text{vac}|\varphi\rangle|^2 \prod_{j=1}^n \varphi(x_j) \quad (6.28)$$

The probability $|\langle\text{vac}|\varphi\rangle|^2$ is a positive functional of φ . It is convenient to write it in the following form,

$$|\langle\text{vac}|\varphi\rangle|^2 = \mathcal{N} \exp -S[\varphi]/\hbar \quad (6.29)$$

where \mathcal{N} is a normalization factor chosen to ensure unit normalization for the Euclidean vacuum:

$$\langle\text{vac}|\text{vac}\rangle = 1 \quad (6.30)$$

The constant \hbar is the usual quantum of action, so $S[\varphi]$ has the dimensions of action; indeed, it will be shown to be a Euclidean version of the action. Note that we have a real positive weighting factor $\exp -S/\hbar$ instead of the oscillatory factor $\exp iS/\hbar$ obtained in Minkowski space.

For the moment, we treat $S[\varphi]$ as a functional which characterises a given field theory. With these definitions, we find:

$$\langle \text{vac} | \prod_{j=1}^n \{\Phi(x_j)\} | \text{vac} \rangle = \frac{\int [d\varphi] \prod_{j=1}^n \{\varphi(x_j)\} \exp -S[\varphi]/\hbar}{\int [d\varphi] \exp -S[\varphi]/\hbar} \quad (6.31)$$

Notice that the result is a *functional average* of $\prod_{j=1}^n \varphi(x_j)$.

The discussion above makes no reference to the origin in function space $\{\varphi\}$. Our choice of operator $\Phi(x)$ was arbitrary; instead, we could have chosen to consider the eigenvalue problem for

$$\tilde{\Phi}(x) = \Phi(x) + f(x)I$$

for any function $f(x)$. Therefore functional integrals should have a translation property

$$\int [d\varphi] F[\varphi + f] = \int [d\varphi] F[\varphi] \quad (6.32)$$

analogous to that for ordinary infinite integrals:

$$\int_{-\infty}^{\infty} dx f(x + a) = \int_{-\infty}^{\infty} dx f(x)$$

When Eq. (6.32) is expanded in f , the linear term is

$$\int [d\varphi] \frac{\delta}{\delta\varphi} F[\varphi] = 0 \quad (6.33)$$

We are now in a position to identify $S[\varphi]$ as an action. In Eq. (6.33), let the functional F be given by the integrand of the numerator of Eq. (6.31):

$$F[\varphi] = \prod_{j=1}^n \{\varphi(x_j)\} \exp -S[\varphi]/\hbar \quad (6.34)$$

Its functional derivative is

$$\frac{\delta F}{\delta\varphi(x)} = \left\{ -\frac{1}{\hbar} \frac{\delta S}{\delta\varphi(x)} \prod_{j=1}^n \{\varphi(x_j)\} + \sum_k \delta^4(x - x_k) \prod_{j \neq k} \varphi(x_k) \right\} \exp -S[\varphi]/\hbar \quad (6.35)$$

If we integrate over φ and use Eq. (6.33), the result is a set of Schwinger-Dyson equations:

$$\langle \text{vac} | \frac{\delta S[\Phi]}{\delta\Phi(x)} \prod_{j=1}^n \{\Phi(x_j)\} | \text{vac} \rangle = \hbar \sum_k \delta^4(x - x_k) \langle \text{vac} | \prod_{j \neq k} \{\Phi(x_k)\} | \text{vac} \rangle \quad (6.36)$$

In the classical limit $\hbar \rightarrow 0$, we find that the variation of S vanishes exactly, as in Eq. (6.21); the contact terms on the right-hand side of Eq. (6.36) are evidently quantum corrections. We conclude that $S[\varphi]$ is the classical action functional, apart from possible $O(\hbar)$ corrections.

It is convenient to summarize results obtained for the set of all Green's functions (6.31) by introducing the generating functional

$$Z[j] = \int [d\varphi] \exp - \left\{ S[\varphi] + \int d^4x j(x)\varphi(x) \right\} \quad (6.37)$$

Variations with respect to the “source” function $j(x)$ result in insertions of the operator $\Phi(x)$:

$$\langle \text{VAC} | \prod_{j=1}^n \{ \Phi(x_j) \} | \text{VAC} \rangle = \frac{1}{Z[0]} \prod_k \left\{ -\frac{\delta}{\delta j(x_k)} \right\} Z[j] \Big|_{j=0} \quad (6.38)$$

For future reference, we note the identity

$$F \left[-\frac{\delta}{\delta j} \right] \exp - \int d^4x j(x)\varphi(x) = F[\varphi] \exp - \int d^4x j(x)\varphi(x) \quad (6.39)$$

which can be verified by expanding the functional F as a Taylor series.

6.3 Evaluating Functional Integrals

So far, we have not tried to interpret the meaning of the “integral over φ ”.

In the lattice approach [55], the coordinates x_μ are restricted to discrete sites $(x_\mu)_n$ so that integrals can be replaced by sums over discrete variables φ_n :

$$\int [d\varphi] = \prod_n \int_{-\infty}^{\infty} d\varphi_n, \quad \varphi_n = \varphi(x_n) \quad (6.40)$$

This approximation introduces a lattice spacing a . Derivatives $\partial\varphi$ are approximated by differences $(\varphi_{n+1} - \varphi_n)/a$. It is assumed that the correct result is obtained in the limit $a \rightarrow 0$.

As first noted by Symanzik [45], the discrete version of a Euclidean path integral has the structure of a partition function in statistical mechanics:

$$Z = \text{Tr} \exp -\beta H \quad (6.41)$$

Consequently, techniques from statistical mechanics can be applied to problems in quantum field theory [53, 55].

The main alternative to the lattice approach is “Gaussian” integration. This leads directly to the rules for Feynman diagrams. As a simple example, we consider a Hermitian Bose field operator $\Phi(x)$ with real eigenvalues $\varphi(x)$.

Let us write each function $\varphi(x)$ as a linear combination

$$\varphi(x) = \sum_n z_n \varphi_n(x) \quad (6.42)$$

of complete orthonormal functions $\varphi_n(x)$ with real coefficients z_n :

$$\begin{aligned} \int d^4x \varphi_m^*(x) \varphi_n(x) &= \delta_{mn} \\ \sum_n \varphi_n(x) \varphi_n^*(y) &= \delta^4(x-y) \end{aligned} \quad (6.43)$$

Note that φ_n may be complex, as in the Fourier analysis of a real function.

The set

$$\{\varphi_n(\mathbf{x}), n = 1, 2, 3, \dots\} \quad (6.44)$$

can be regarded as defining orthogonal vectors with unit length in function space. As the parameters z_n are varied over all real values, all functions φ are produced. Thus

$$\{z_1, z_2, z_3, \dots\}$$

is a set of coordinates for an infinite-dimensional Cartesian space, where each point labels a function φ . The measure for path summation is given by a volume element in $\{z_n\}$ space:

$$[d\varphi] = \prod_n \left(dz_n / \sqrt{2\pi} \right) \quad (6.45)$$

This prescription does not depend on the choice of complete set (6.44), provided that boundary conditions defining the function space $\{\varphi\}$ are respected. The normalization $1/\sqrt{2\pi}$ for each z integral is a convention; such factors cancel in the expression (6.31) for Euclidean Green's functions.

The standard Gaussian integral is

$$\mathcal{I} = \int [d\varphi] \exp -\frac{1}{2} \int d^4\mathbf{x} \varphi(\mathbf{x}) D_{\mathbf{x}} \varphi(\mathbf{x}) \quad (6.46)$$

where $D_{\mathbf{x}}$ is a differential operator. It is convenient to choose the eigenfunctions of $D_{\mathbf{x}}$ as a basis (6.44) for function space:

$$D_{\mathbf{x}} \varphi_n(\mathbf{x}) = E_n \varphi_n(\mathbf{x}) \quad (6.47)$$

Then Eq. (6.46) becomes a product of ordinary Gaussian integrals, each of the form

$$\int_{-\infty}^{\infty} dz \exp -\frac{1}{2} E z^2 = \sqrt{2\pi/E}$$

So we find

$$\mathcal{I} = \prod_n \left\{ \int_{-\infty}^{\infty} \left(dz_n / \sqrt{2\pi} \right) \exp -\frac{1}{2} E_n z_n^2 \right\} = (\det D)^{-1/2} \quad (6.48)$$

where the determinant of the operator $D_{\mathbf{x}}$ is defined to be the product of its eigenvalues:

$$\det D = \prod_n E_n \quad (6.49)$$

This example can be easily generalized to include a source $j(\mathbf{x})$ for φ :

$$\mathcal{I}[j] = \int [d\varphi] \exp - \int d^4\mathbf{x} \left\{ \frac{1}{2} \varphi(\mathbf{x}) D_{\mathbf{x}} \varphi(\mathbf{x}) + j(\mathbf{x}) \varphi(\mathbf{x}) \right\} \quad (6.50)$$

A shift of integration variable

$$\varphi \longrightarrow \varphi + D_{\mathbf{x}}^{-1} j \quad (6.51)$$

yields the result

$$\begin{aligned}\mathcal{I} &= (\det D)^{-1/2} \exp \frac{1}{2} \int d^4x j(x) D_x^{-1} j(x) \\ &= (\det D)^{-1/2} \exp \frac{1}{2} \int d^4x \int d^4y j(x) G(x, y) j(y)\end{aligned}\quad (6.52)$$

where

$$G(x, y) = \sum_n E_n^{-1} \varphi_n(x) \varphi_n^*(y) = D_x^{-1} \delta(x - y) \quad (6.53)$$

is the propagator of the operator D_x .

The identity (6.39) permits an immediate generalization of eq. (6.53) to include a potential $V[\varphi]$:

$$\int [d\varphi] \exp - \int d^4x \left\{ \frac{1}{2} \varphi(x) D_x \varphi(x) + j\varphi + V[\varphi] \right\} = \exp - \int d^4x V[-\delta/\delta j(x)] \mathcal{I}[j] \quad (6.54)$$

These expressions can be used to generate Feynman rules very efficiently; (indeed, that is essentially how Feynman arrived at them originally). For example, consider the Euclidean version of the theory (5.38):

$$S[\varphi] = \int d^4x \left\{ \frac{1}{2} \varphi(-\partial^2 + m^2)\varphi + \lambda \varphi^4 \right\} \quad (6.55)$$

Then, according to Eq. (6.54), the generating functional $Z[j]$ of (6.37) satisfies the formula

$$Z[j]/Z[0] = \exp \left\{ -\lambda \int d^4x (\delta/\delta j(x))^4 \right\} \exp \frac{1}{2} \int d^4x \int d^4y j(x) G(x - y) j(y) \quad (6.56)$$

where

$$G(x - y) = (-\partial^2 + m^2)_x^{-1} \delta^4(x - y) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp -ik \cdot (x - y)}{k^2 + m^2} \quad (6.57)$$

is the Euclidean Feynman propagator. In Feynman diagrams, $G(x - y)$ is represented by a line joining x and y . The second exponential factor in (6.56) provides any number of propagator lines, which are then joined at their ends by the action of the first exponent. Each time the exponent $-\lambda \int d^4x (\delta/\delta j(x))^4$ acts, a four-legged vertex is formed. Feynman diagrams are staple fare in textbooks, so there is little point trying to cover them further in these brief notes.

There are two main problems with the formalism developed above. First, there may be zero modes $E_k = 0$ which prevent D_x from being inverted, especially if the zero eigenvalue is discrete. In that case, the corresponding integrals over z_k must be isolated and handled separately via the method of “collective coordinates” [56]. For continuous eigenvalues, it depends on questions of measure. For example, the operator $D_x = -\partial^2$ has eigenfunctions

$$u_q(x) = \exp -iq \cdot x \quad (6.58)$$

labelled by the continuous four-vector q_μ , with eigenvalue q^2 . There are degenerate zero modes $q^2 = 0$, but the propagator

$$D_x^{-1} \delta^4(x - y) = \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2} \exp -iq \cdot (x - y) = \frac{1}{4\pi^2(x - y)^2} \quad (6.59)$$

is well defined. On the other hand, gauge theories have an enormous symmetry which produces a functionally degenerate set of zero modes. A propagator can be constructed only if contributions from zero modes are restricted by fixing the gauge sufficiently.

The other problem concerns ultra-violet infinities, which have to be renormalized. For example, products \prod_n and sums \sum_n may blow up for large eigenvalues E_n , $n \rightarrow \infty$. The product (6.49) defining $\det D$ is invariably divergent. One way of controlling it is to use the “zeta-function” method [57]. It makes the replacement

$$\prod_n E_n = \exp \sum_n \ln E_n \longrightarrow \exp -\frac{\partial}{\partial s} \sum_n E_n^{-s} \quad (6.60)$$

where s is a complex regulator. The resulting sum converges for $\text{Re } s$ sufficiently large,

$$\sum_n E_n^{-s} = \text{Tr} D^{-s} = \zeta_D(s) \quad (6.61)$$

so $\zeta_D(s)$ can be continued analytically to $s = 0$:

$$(\det D)_{\text{renorm}} = \exp -\zeta'_D(0) \quad (6.62)$$

Also, the operation $V[-\delta/\delta j(x)]$ in (6.54) can generate ultra-violet infinities at short distances by causing too many variations $\delta/\delta j(x)$ to act at the same point. Loop diagrams in perturbation theory require renormalization because of this problem.

6.4 Fermionic Integration

According to Eq. (5.91), Euclidean fermion operators *anti*-commute throughout Euclidean space. Since they do not commute, the method of simultaneous diagonalization used for bosons in Eq. (6.22) is not immediately applicable.

This problem is circumvented by a trick [48]. We regard the eigenvalue problem as being formulated for operators $\Psi(x)$, $\bar{\Psi}(x)$ and eigenvalues $\psi(x)$, $\bar{\psi}(x)$ which take *Grassmann* values. (Grassmann numbers $a_1 \dots a_\ell$ are nilpotent and anticommute with each other: $a_m^2 = 0$, $a_m a_n = -a_n a_m$.) If the eigenvalues anticommute with each other, simultaneous diagonalization is permitted,

$$\Psi(x)|\psi, \bar{\psi}\rangle = \psi(x)|\psi, \bar{\psi}\rangle, \quad \bar{\Psi}(x)|\psi, \bar{\psi}\rangle = \bar{\psi}(x)|\psi, \bar{\psi}\rangle \quad (6.63)$$

where the states $|\psi, \bar{\psi}\rangle$ are not Grassmann valued, and are assumed to span the Euclidean state vector space:

$$\int [d\bar{\psi} d\psi] |\psi, \bar{\psi}\rangle \langle \psi, \bar{\psi}| = I \quad (6.64)$$

A Gaussian interpretation of the sum over fermionic paths in Eq. (6.64) (sometimes called a “Gaussian” integral) is possible for integrals of Grassmann-valued functionals of $\psi, \bar{\psi}$.

First, we expand the eigenvalue functions in terms of a complete orthonormal set of ordinary spinor functions $u_n(x)$:

$$\psi(x) = \sum_n \xi_n u_n(x), \quad \bar{\psi}(x) = \sum_n \eta_n u_n^\dagger(x) \quad (6.65)$$

The Grassmann character of $\psi(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ is carried by the coefficients $\{\xi_n, \eta_n\}$, which are independent Grassmann variables.

We have to interpret an integral of the form

$$[d\bar{\psi}d\psi] = \prod_n (d\eta_n d\xi_n) \quad (6.66)$$

where the differentials $d\eta_n$ and $d\xi_n$ are also Grassmannian. This integral should be translation invariant, as in the bosonic case (6.32), so each Grassmann integral must also have this property,

$$\int d\xi f(\xi + \xi_0) = \int d\xi f(\xi) \quad (6.67)$$

where ξ_0 is an independent Grassmann constant. Since ξ is nilpotent, the Taylor series for $f(\xi)$ terminates,

$$f(\xi) = f(0) + \xi f'(0) \quad (6.68)$$

so it is sufficient to determine the integrals of ξ and of a constant. So, choosing our function to be $f(\xi) = \xi$,

$$\int d\xi \xi = \int d\xi \xi + \int d\xi \xi_0 \quad (6.69)$$

we conclude that the only consistent answer for the integral over a constant is zero:

$$\int d\xi = 0 \quad (6.70)$$

The only remaining integral, that over ξ , must give an ordinary non-zero number which, by convention, is chosen to be unity

$$\int d\xi \xi = 1 \quad (6.71)$$

so that $\int d\xi f(\xi)$ is just the derivative $f'(0)$ at the origin. Whatever normalization is chosen here, it will cancel out in the ratio (6.31).

The fermionic Gaussian integral analogous to (6.46) can now be evaluated directly:

$$\begin{aligned} \mathcal{I} &= \int [d\bar{\psi}d\psi] \exp - \int d^4\mathbf{x} \bar{\psi}(\mathbf{x}) D_{\mathbf{x}} \psi(\mathbf{x}) \\ &= \prod_n \left\{ \int \int d\eta_n d\xi_n \exp -E_n \eta_n \xi_n \right\} \\ &= \prod_n \left\{ \int \int d\eta_n d\xi_n (1 - E_n \eta_n \xi_n) \right\} \\ &= \prod_n E_n = \det D \end{aligned} \quad (6.72)$$

Note that the answer is $\det D$ instead of $(\det D)^{-1/2}$ for real boson fields (Eq. (6.48)), or

$$(\det D)^{-1} = \int [d\varphi][d\varphi^\dagger] \exp - \int d^4\mathbf{x} \varphi^\dagger(\mathbf{x}) D_{\mathbf{x}} \varphi(\mathbf{x}) \quad (6.73)$$

for complex boson fields. For Majorana fermions (mentioned in sec. 5.2), the result is $(\det D)^{1/2}$.

A generating functional analogous to (6.37) can be constructed if Grassmann sources $\chi, \bar{\chi}$ are introduced for the fermion fields $\bar{\psi}, \psi$:

$$\begin{aligned} Z[\chi, \bar{\chi}] &= \int [d\bar{\psi}d\psi] \exp - \int d^4x \{ \bar{\psi}(x) D_x \psi(x) + \bar{\chi}\psi + \bar{\psi}\chi \} \\ &= \det D \exp \int d^4x \int d^4y \bar{\chi}(x) G(x, y) \chi(x) \end{aligned} \quad (6.74)$$

Here the Green's function $G(x, y)$ is given by

$$G(x, y) = \sum_n E_n^{-1} u_n(x) u_n^\dagger(y) = D_x^{-1} \delta(x - y) \quad (6.75)$$

instead of Eq. (6.53). The Grassmann character of χ and $\bar{\chi}$ is essential; otherwise, the antisymmetry of fermionic Euclidean operators inside Green's functions (5.90), as required by Eq. (5.91), would not be reproduced.

The appropriate analogue of (6.54) allows interactions to be introduced and hence Feynman rules to be deduced.

It may seem from this discussion that fermionic integration is not more problematic than the bosonic version. Certainly, perturbation theory with fermions is as straightforward as that for bosons. However the principles behind fermionic integration are somewhat obscure. The connection between completeness and the sum over functions is clear for bosons, but how are we to understand Grassmann character in the fermionic completeness relation (6.64)?

These questions may be of practical importance in non-perturbative calculations. Certainly, it is recognised that it is difficult to put fermions on the lattice [55]. There is clearly a need for a better understanding of fermionic integration.

7. Outlook

At this stage, I could have continued with a lightning introduction to gauge theories, Fadde'ev-Popov ghosts, and Becchi-Rouet-Stora invariance, as in the verbal version of these notes. In the event, it became evident that a rushed version of these topics would not be of great use here.

Indeed, I fear that these lectures have sacrificed pedagogy for brevity. What was presented was just a rough outline, a framework on which a satisfactory understanding of quantum field theory might be developed. For example, I have neglected calculational techniques completely. My excuse is that such skills take many lectures to teach, and are covered at great length in any textbook of note.

A glance at the Table of Contents of such textbooks should provide readers with a list of topics to pursue. A typical list could include the following: systematic use of Feynman rules for any local Lagrangian, calculation of low-order amplitudes in perturbation theory, calculation of rates and cross-sections, regularization, renormalization and the renormalization group, lattice approximation and

connection with statistical mechanics, bound-state problems, exact and approximate symmetries, anomalies, gauge theories, quantum chromodynamics and the standard model, supersymmetry, field-theoretic model building, grand unification, and inevitably, strings.

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